1. Consider the problem

\[ \varepsilon \frac{dy}{dx} + y = x, \quad x > 0; \quad y(0) = 1, \]

with \( \varepsilon \ll 1 \). Find and match two terms of the outer and inner expansions of the solution \( y(x) \). Write down the two-term composite solution.

2. An elastic beam under tension is called a beam spring. If such a beam is positioned horizontally and its ends are clamped at the same height, the equations governing the displacement \( u(x) \) of the beam spring can be nondimensionalized to yield

\[ \varepsilon \frac{d^4 u}{dx^4} - \frac{d^2 u}{dx^2} = -p(x) \]

\[ u(0) = \frac{du}{dx}(0) = 0, \quad u(1) = \frac{du}{dx}(1) = 0, \]

where \( p(x) \) is an applied external loading (measured upwards) and \( \varepsilon \) is a relative measure of the beam’s rigidity in comparison to the tension. Consider a beam spring of small rigidity \( \varepsilon \ll 1 \) with a uniform external loading \( p(x) \equiv 2 \). The symmetry of the problem suggests that there is a boundary layer located at each end of the interval. Notice the boundary conditions suggest that \( u \) may be a small quantity in the boundary layers. For this reason, the naive expansion is not an appropriate expansion for \( u \) in the layers. The outer solution is in turn affected, and consequently, the naive expansion is not appropriate for the outer solution either.

(a) Expand the outer solution as

\[ u_{\text{out}}(x) \sim U_0(x) + \sqrt{\varepsilon} U_1(x) + \cdots \]

and find \( U_0(x) \) and \( U_1(x) \).

(b) Determine the appropriate stretching coordinate \( \xi \) in the boundary layers, expand the inner solutions as

\[ u_{\text{in}}(\xi) \sim \sqrt{\varepsilon} U_1(\xi) + \varepsilon U_2(\xi) + \cdots, \]

and find \( U_1(\xi) \) and \( U_2(\xi) \).

(c) Form a uniformly valid two-term composite expansion of \( u(x) \).

3. The problem

\[ x^3 \frac{du}{dx} = \varepsilon [(1 + \varepsilon) x + 2x^2] u^2, \quad 0 \leq x \leq 1; \quad u(1) = 1 - \varepsilon \]

involves a two-ply boundary layer, and therefore, the asymptotic solution requires three expansions: an outer, an intermediate, and an inner.

(a) Find three nonzero terms of the outer solution expansion and determine the region of nonuniformity.

(b) The results of part (a) indicate the location and the width of the boundary layer. Stretch the spatial coordinate accordingly and find three nonzero terms of the intermediate solution expansion. It is convenient to match after each term before proceeding to the next order. Determine the region of nonuniformity.

(c) Stretch the spatial coordinate according to the results of part (b). Find two nonzero terms of the inner expansion. Again, it is convenient to match the first nonzero term before finding the second.

(d) Show that the composite expansion of \( u \) is given by

\[ u_{\text{comp}}(x) \sim \frac{x^2 + \varepsilon^4 x}{(x + \varepsilon)(x + \varepsilon^2)} + O(\varepsilon^3). \]

Though it is not required as part of the assignment, it is of interest to plot the three expansions found in parts (a)–(c) to visualize the matching in regions where the domains of validity of the various expansions overlap. It is also worthwhile to plot the the composite expansion found in part (d).