Resolution of Poisson Paradox

\[ \langle \beta_+ \rangle = \frac{2}{\lambda} = \text{avg length of time interval containing a given point (t)} \]

length-biased sampling →

\[ \frac{1}{\Delta t} = \text{avg time interval with each interval weighted the same.} \]

\[ \downarrow \]

[Diagrams of time intervals with varying lengths]
\[ p_T(t) = (1)e^{-\lambda t} \]

Length-biased \[ p(t) \propto e^{-\lambda t} \]

\[ p(t) = \lambda^2 t e^{-\lambda t} \] (??)

Easy example: \( T = \begin{cases} 1 & \text{prob } \frac{1}{2} \\ 1/2 & \text{prob } \frac{1}{2} \end{cases} \)

Avg over events: \( \langle T \rangle = \frac{1}{2}(1) + \frac{1}{2}(1/2) = \frac{3}{4} \)

Avg over intervals containing a given time \( t \):
\[ \langle T \rangle_t = \frac{2}{3} \langle 1 \rangle + \frac{1}{3} \langle 1/2 \rangle = \frac{5}{6} > \frac{3}{4} \]
What about calculating \( N(t), \delta, \delta, \beta \) for general renewal processes?

Renewal argument (K+T Sec. 5.4 A)
- First-step analyses for renewal processes

Consider \( M(t) = |E N(t) | \)

\[ |E N(t) | = \int_0^\infty \left|E \left[ N(t) \mid T_i = t \right] \right| dF(t) \]

(conditioning on time \( T_i \) of first event)

\[ (dF(t) = p(t) \, dt) \]

(\( p(t) \) is the PDF)

(\( \text{and EXP formula for continuously distributed} \))
\[ E[N(t) | T = t'] = 1 + E[N(t - t')] \]

For \( 0 \leq t' \leq t \)

Important that \( t' \) is beginning of renewal period so equivalent to starting over

\[ = 0 \text{ if } t' > t \]

\[ M(t) = \int_0^t (1 + M(t - t'))dF(t') \]

\[ = F(t) - F(0) + \int_0^t M(t - t')dF(t') \]

\[ M(t) = F(t) + \int_0^t M(t - t')dF(t') \]

-Example of "renewal equation"

-Can use similar argument for other moments of \( N(t) \) (HW)
How solve renewal equation?
- Laplace transform

\[ \hat{M}(z) = \int_0^\infty e^{-zt} M(t) \, dt \]

where \( z \in \mathbb{C} \) with large enough real part,

\[ M(z) = \hat{F}(z) + \int_0^\infty e^{-zt} \, dF(t) \]

\[ \hat{M}(z) = \frac{\hat{F}(z)}{1 - z \hat{F}(z)} \]

Invert Laplace transform:

\[ M(t) = \frac{1}{2\pi i} \int_C e^{zt} \hat{M}(z) \, dz \]

\( C \): Bromwich contour
If $F(z)$ is simple enough
then can solve explicitly
using tables, complex integration
theory.

If not... can still usually
derive the $z \to \infty$ asymptotics
- contour integration theory;
usually residue of
the singularity of $F(z)$
that is farthest to
the right (largest real part)
Renewal Theorem (K+T Sec. 5.4 + 5.5)

Let $F(t)$ be a nondecreasing fn with $F(0) = 0$, $\lim_{t \to \infty} F(t) = 1$

and $F$ is not arithmetic

(points of increase don't fall on a lattice containing 0)

($F$ is a CDF not concentrated on a lattice containing 0)

1) For any bounded function $a(t)$ the integral eqn

\[ A(t) = a(t) + \int_0^t A(t-t')dF(t') \]

has unique soln bounded on finite intervals,
2) The solution to \( a(t) \) can be expressed as

\[
A(t) = a(t) + \int_0^t a(t-t') dM(t')
\]

where \( M(t) \) is unique soln to

\[
M(t) = F(t) + \int_0^t M(t-t') dF(t)
\]

3) If \( a(t) \in L^1 \left( \int_0^\infty |a(t)| dt < \infty \right) \)

then

\[
lim_{t \to \infty} A(t) = \int_0^\infty a(t-t') dt \quad \text{(constant)}
\]

where \( M = \int_0^\infty (1 - F(t)) dt \)

\[
= \langle T \rangle
\]
(4) (Corollary)

\[ \lim_{t \to \infty} M(t) - M(t-s) = \frac{s}{M} \]

Application to calculation of statistics of residual life \( \Delta_t \)

\[ A_s(t) = \text{Prob}(\Delta_t > s) \]

Renewal argument:

\[ \text{Prob}(\Delta_{t'} > s \mid T_t = t') = \begin{cases} 1 & \text{if } t' > t+s \\ 0 & \text{if } t' < t+s \end{cases} \]

\[ = \text{Prob}(\Delta_{t-t'} > s) \quad \text{if } 0 < t < t' \]
$L' > t + s$

$T = T'$
\[ A_s(t) = \text{Prob}(\hat{T} > s) \]
\[ = \int_0^\infty \text{Prob}(\hat{T} > s | T_1 = t^-) \, dF(t^-) \]
(\text{and exp formula})
\[ = \int_0^t A_s(t^-) \, dF(t^-) + \int_t^{t+s} 1 \, dF(t^-) \]
\[ = \int_0^t A_s(t^-) \, dF(t^-) + \Theta \]
\[ \lim_{t \to \infty} F(t^-) = F(t + s) \]
\[
A_s(t) = 1 - F(t+s) + \int_0^t A_s(t-t')dF(t')
\]

renewal equation

\[
\lim_{b \to \infty} A_s(t) = \int_0^\infty (1 - F(t+s))d\xi
\]

\[
\lim_{s \to \infty} \text{Prob}(\xi > s) = \int_s^\infty (1 - F(t+s))d\xi
\]

provided \( \mu < T < \infty \)

Similarly (K+T Sec 5.6 a)

\[
\lim_{t \to \infty} \text{Prob}(\delta_t > s) = \int_s^\infty (1 - F(t+s))d\xi
\]

\[
\lim_{t \to \infty} \text{Prob}(\beta_t > s) = \frac{1}{\mu} \int_s^\infty \xi dF(t) \quad \text{(length-biased)}
\]
How do the PDF's for these variables help in applications?

\[ N(t) \]

excess/residual life: signaling problems

(14.1 Sec. 5.3)

Continuous-time branching process

- renewal can beat volatile renewal plan assumptions

- replace x form

Inventory models: Section 5.8 C

Prob of avoiding bankruptcy of insurance company: Section 5.3 C