04/17/06 Long-Time Properties of CTMC, Renewal Processes

Transience vs. Recurrence?

- Communication classes same as for discrete time case.
- Recurrence/transience determined by looking at associated DTMC.

Define prob trans matrix \( \tilde{P} \) for associated DTMC

\[
\tilde{P}_{ij} = \frac{A_{ij}}{A_i}
\]
One more modification:

For absorbing state ($A_i = 0$),

\[
\tilde{P}_{ij} = \delta_{ij} \text{ for absorbing state } i,
\]

Transience/recurrence for CTMC

\[\Longleftrightarrow\] tranisience/recurrence for associated DTMC,

Positive Recurrence vs. Null Recurrence

As for discrete time case,

positive recurrence \[\Longleftrightarrow\] existence of stationary distribution

For closed commun class,

But stat. dist. CTMC \[\neq\] stat. dist. associated DTMC,
To calculate static dist. for CTMC:
\[ \pi_i = \text{Prob} (X(t) = i) \]

For any prob dist (need not be stationary)
Kolmogorov forward equs
\[ \frac{d \vec{\pi}}{dt} = \vec{\pi} A \]

for any prob dist \( \vec{\pi} \).

Stationary dist.
\[ \frac{d \vec{\pi}}{dt} \to \vec{0} \]

\[ \Rightarrow \quad \vec{\pi} A = \vec{0} \]

and
\[ \sum_j \pi_j = 1 \]
\[ \forall j \in S \quad \pi_j > 0 \text{ for all } j \in S \]

Lawler Ch. 3

Resnick Sec. 5.5

Detailed balance ideas still apply.
One more question: Accumulated cost/reward
- absorption probabilities calculated from associated DTMC

Cheap derivation of formula for accumulated cost/reward from transient states
- Properly done by first-step analysis
  Lawler Ch. 3, Resnick S. 5.5
- Kolmogorov backward equ
  generalizes to SDE

Suppose we have a transient class $T$
For each jet, one collects cost/reward
at rate $f(c_j)$
Consider discretizing the CTMC into equally spaced times $t_i$. The resulting DTMC has $\hat{P} = e^{-A\Delta t}$ and cost/reward per epoch $f(j)\Delta t$ in state $j$.

The expected cost/reward for CTMC:

$$w_i = \mathbb{E} \left[ \int_0^T f(X(t)) dt \mid X(0) = i \right]$$

and $T = \inf \{ t \geq 0 : X(t) \notin \mathcal{T} \}$

Applying discrete-time formula for discretized version

$$ (I - Q) \overrightarrow{w} = f \Delta t $$
\[
\frac{(I - \alpha)}{\Delta t} w = f
\]

\[
\alpha = e^{B \Delta t} \text{ where } B \text{ is the submatrix of } A \text{ corresponding to transient states } \mathbb{T}
\]

\[
\lim_{\Delta t \to 0} \frac{I - e^{B \Delta t}}{\Delta t} w = f
\]

\[
-B w = f
\]

Cost/reward formula for CTMC,

Also written as:

\[
-A w = f
\]

where \( w_j = 0 \) and \( f_j = 0 \)

for \( j \notin \mathbb{T} \)
Technical problems with proofs:
- neglect transitions within $A$, $t$
- at the random time to leave $T$
  - could fall between time steps,
- approximated states as unchanged during $dt$ for calculating reward.

Examples: Lawler Sec. 3.3
Ko Ch. 4

Optimal stopping theory: Lawler Ch. 4
Renewal Processes

- point process on positive real line

\[ t = 0 \]

\[ \{ T_j \}_{j=1}^{\infty} \]

\[ \beta_t = \delta_t + \delta_t^* \text{ total life} \]

\[ N(t) \]

\[ \delta_t = \text{current life} \]

\[ \delta_t^* = \text{residual life} \]

Such that \( T_1, T_2 - T_1, T_3 - T_2, \text{etc.} \)

are iid rvs with CDF

\[ F(t) = \text{prob} \left( T \leq t \right) \]

time between points
For $P(t) = 1 - e^{-\lambda t}$ for $t \geq 0$

$= 0$ for $t = 0$

then get Poisson point process

with intensity $\lambda$.

Renewal counting process

$N(t) = \# \text{ points in } (0, t]$

-not a Markov process unless

$P(t)$ is as above (e.g. drift.)

Examples: A) Poisson process
B) Detectors or neurons w/ refractory period
C) Following distances in traffic
D) Equipment replacement times
E) Inventory demand + restocking
P) Distributing time between special events in Markov chains
- successive visits to a state
- times of successive
  maxima
Revisit Poisson point process

With intensity $\lambda$

$\delta_t$: From Kolmogorov backward analysis

CDF: $F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

Exp distributed with mean $\frac{1}{\lambda}$.

$\delta_t$: Time-reversed ME

$\text{Poisson counting process}$ (detasct balance)

$\text{Prob}(\delta_t > t') = \text{Prob}(\text{no events in } [t-t', t])$

$= e^{-\lambda t'}$ for $0 < t' \leq t$

$\text{Prob}(\delta_t > t') = 0$ for $t' > t$

CDF: $F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{for } 0 \leq t \leq t \\ 1 & \text{for } t' > t \\ 0 & \text{for } t' < 0 \end{cases}$
Expected residual life = \( \mathbb{E}[\delta_t] = \frac{1}{\lambda} \)

Expected current life =

\[ \mathbb{E}[\delta_t] = \int_{-\infty}^{\infty} t \, dF_{\delta_t}(t) \]

\[ = \int_{\delta_t}^{\infty} (1 - F_{\delta_t}(t)) \, dt \]

int by parts

generally true for non-negative rvs

\[ = \int_{0}^{\infty} e^{-\lambda t'} \, dt' = \frac{1 - e^{-\lambda t}}{\lambda} \]
Expected Total Life:

\[ \langle \beta_t \rangle = \langle 0_t + \delta_t \rangle = \frac{2 - e^{-\lambda t}}{\lambda} \]

For large time: \( \langle \beta_t \rangle \approx \frac{2}{\lambda} \)

But by strong Markov property,

expected time spent in a
state is \( \frac{1}{\lambda} \) (numerical sim.

Poisson paradox)