5/13/06 Numerical Simulation and Long Time Inference of CTMC and Poisson process

HW 4 posted, due May 3 (Wed) 5 PM (AE 301 or 310)
- all late submissions May 5 5 PM
- Final exam May 8

Numerical simulation of Poisson point process

Subdivide into "convenient" domains with finite volume $\mathcal{A}_j$
For each $A_j$, generate $N(\lambda \text{Vol}(A_j))$

as a Poisson r.v. with mean $

\lambda \text{Vol}(A_j)$

Poisson: $\text{Prob}(X = j) = e^{-\mu} \frac{\mu^j}{j!}$

with mean $\mu$ for $j = 0, 1, 2, \ldots$

Can do this by binary digitization on

$[0, 1) \bigcup \{1/2, 1/4, \ldots\}$ simulate Poisson r.v.

Alternatively: Simulate exp dist r.v. $\{T_j\}$

with PDF $p_T(t) = Me^{-Mt}$

Keep simulating $T_j$ until

$\leq T_j > 1$. Then

$\sum_{j=1}^{n} \text{Sim} \geq n-1$ will be Poisson r.v. with

mean $\mu$. \[\]
This derives from connecting a Poisson distribution to a 1-D Poisson point process.

\[ T_1 \leq T_2 \leq T_3 \leq T_4 \]

Intensity = \( \mu \); any time between events is \( \text{exp} \) distributed with mean \( \frac{1}{\mu} \).

The points in \([0,1]\) is Poisson r.v. with mean \( \mu \).

Back to simulating multi-D Poisson point processes.

Once we know \( N(A_j) \), the points are distributed uniformly in \( A_j \) and independently by the rejection method.
Why use Poisson point processes in modeling?
- fairly simple
- Poisson limit theorems
  - like Central Limit Theorems for Gaussian r.v.s.

Central Limit Theorem: The cumulative effect of many independent small uncertainties is Gaussian.
Rayleigh–Bénard convection

A. Libchaber

Poisson limit theorem; Karlin + Taylor Sec. 5.9

The cumulative effect of many rare independent renewal processes is a Poisson point process.
- cooperation between renewal processes ⇒ rain Poisson assumption
- externally driven fluctuations
  - inhomogeneous Poisson point process
- clumping
  - compound Poisson point process

\[ X \times X \times X \]
Markov Times and Strong Markov Property

- Remark Sec. 1.8 (discrete time)
- K+T Ch. 6 (continuous time)

Given a stochastic process \( X(t) \), define \( \mathcal{A}_t \) as the \( \sigma \)-algebra of events that can be decided up to time \( t \).

For separable (continuous) process \( X(t) \) w/ associated prob. space \( \Omega \), \( \mathcal{A}_t \) is generated by sets
\[
A = \{ w \in \Omega : X(s_i, w) \in B_i \text{ for } i = 1, \ldots, n \}
\]
and \( 0 \leq s \leq t \).

The \( \sigma \)-algebra \( \mathcal{F}_t \) is called a filtration if \( \mathcal{F}_t \subseteq \mathcal{F}_s \) for \( t \leq s \).

Markov property:

\[
\Pr(X(t) = j \mid \mathcal{F}_s) = \Pr(X(t) = j \mid X(s))
\]

for \( 0 \leq s \leq t \).

What if the future and past is separated by a random time \( s \)?
Markov time (stopping time) $\tau$
is a random variable defined
on prob space $\Omega$ s.t.,

$\{ \tau \leq t \} \in \mathcal{A}_t$ for all $t \geq 0$.

That means that given info
about $X(t)$ over $[0,t]$,
you can tell whether or not
event corresponding to random
time $\tau$ occurred by $t$.

Examples
- first hitting time for a state
- third hitting time
- time to accumulate certain reward
- time at which a CTMC
  changes state.
Strong Markov property:

For Markov times $\tau$:

$$\text{Prob}(X(t+\tau) = j \mid \sigma_\tau)$$

$$= \text{Prob}(X(t+\tau) = j \mid X(t))$$

for $t > 0$.

- generalizes Markov property to random times

CTMC have strong Markov property if right-continuous,

- Friedman SDE

Sec. 2.2
Simulate CTMC:
- 1) generate random time to stay in a state
- 2) generate next state after jump.

1) Suppose $X(s) = i$. Let

$$T = \inf \{ t \geq 0 \mid X(t+s) \neq i \}$$

$$\text{Prob} \left( T > t \mid X(0) = i \right) = e^{-A_i t}$$

where $\bar{A}_i = \sum_{j \neq i} A_{ij}$, trans. rates $i \to j$

Same argument as Poisson process
2) Let \( X \) denote states immediately before and after a jump, then

\[
\text{Prob}(X_+ = j \mid X_- = i) = \frac{A_{ij}}{A_i}.
\]

\[A_{12} = 1, \quad A_{13} = 4\]
Summary of Numerical Schemes

1) Given that one starts in state i, generate time T until next jump happens by exp dist. rv. with mean \( \frac{1}{A_i} \)

2) At the jump go to state j with probability \( \frac{A_{ij}}{A_i} \)

Repeat...

Rigorous justification: Karlin+Taylor Ch. 14
This numerical scheme is equivalent to the following:

Start in state \( i \).

Calculate \( 0 \) random times \( \{ T_{ij} \}_{j \neq i} \) according to exp dists with

\[
\text{mean } \frac{1}{A_{ij}} \quad (\text{for all } j: A_{ij} \neq 0)
\]

Make the transition according to the earliest of these times.