We allow finite or countably many states, but now the time parameter is continuous.

State of system: $\mathbf{X}(t)$

where $t \in \mathbb{R}$ need not be integer.

Convention:

left-continuous trajectories
Markov property in continuous time:

\[
\Pr(X(t) = j \mid X(t_1) = i_1, X(t_2) = i_2, \ldots, X(t_n) = i_n) = \Pr(X(t) = j \mid X(t_n) = i_n)
\]

when \( t_1 < t_2 < \ldots < t_n < t \)

(can be generalized by considering limits and using left-continuity so that predicting the future given the past, only depends on last moment of the past.)

Time-homogeneous MC

\[
\Pr(X(t) = j \mid X(s) = i) = P_{ij}(t-s) \quad \text{for } ij, s < t
\]
Discrete vs. Continuous-Time MCs

Observing a continuous-time MC at regular intervals gives a discrete-time MC

\[ X_n = X(n \Delta t) \]

So it that is all one cares about, discrete-time MC may be best.

When would one want to use a continuous-time MC model?

1) Important to observe all state transitions (and maybe precise timing), i.e., catastrophic state,
2) Often it is easier to describe the instantaneous rates of change of variable by considering processes acting on it:

- Physics, chemistry, engineering
- Mechanical forces
- Atomic transition rates
- Also useful for phenomenological models
- Different effects usually act additively on rates of change but not to changes over finite time.
3) Sensible, esp. for continuous-space processes w/ natural time step at.

4) Continuous-time MC can be easier to simulate!

Mathematical Formulation for Time-Homogeneous Continuous-Time MC

Transition Probability Function:

\[ P_{ij}(t) = \text{Prob} (X(t+t')=j \mid X(t)=i) \]

for \( t > 0 \)

Encode its info in terms of transition rates.
Taylor expansion about \( t = 0 \):

\[
P(t) = I + At + o(t)
\]

\\[\text{matrix} \quad \uparrow \quad \text{identity} \quad \uparrow \quad \text{higher order terms} \]

\[
\frac{o(t)}{t} \to 0 \quad \text{as} \quad t \to 0
\]

Existence of first derivative

\[
A = \lim_{t \to 0} \frac{P(t) - I}{t}
\]

- see Karlin & Taylor;
- technicalities when \( \infty \) states,
- continuous time,

\[A\] \text{ is the infinitesimal}

\text{generator of the}

\text{continuous-time MC,}

- aka transition rate matrix
- encodes the dynamics of the MC.
$P_{i1}(t)$, $P_{10}(t)$

Entries of infinitesimal generator:

Off-diagonal ($i \neq j$)

$$A_{ij} = \lim_{\Delta t \to 0} \frac{\text{Prob}(X(t+\Delta t) = j | X(t) = i)}{\Delta t} \geq 0$$

= rate of transition $i \to j$
Diagonal entry \((i = j)\)

\[
A_{ii} = \lim_{\Delta t \to 0} \frac{\text{Prob}(X(t + \Delta t) = i \mid X(t) = i) - 1}{\Delta t}
\]

\[
= - \lim_{\Delta t \to 0} \frac{\text{Prob}(X(t + \Delta t) \neq i \mid X(t) = i)}{\Delta t}
\]

= minus total transition rate out of state \(i\)

\[
A_{ii} = - \sum_{j \neq i} A_{ij} \quad \text{because}
\]

\[
\text{Prob}(X(t + \Delta t) \neq i \mid X(t) = i)
\]

\[
= \text{Prob}(X(t + \Delta t) \in S \setminus \{i\} \mid X(t) = i)
\]

Equivalently, \(\sum_{j \in S} A_{ij} = 0\) for all \(i \in S\)
To define dynamics of continuous-time MC:

1) Model transition rates $i \to j$, enter in $A_{ij} \geq 0$ for $j \neq i$

2) Set $A_{ii} = - \sum_{j \neq i} A_{ij}$ for all $i \in S$

3) Check that technical conditions are satisfied (Karlin + Taylor)

Then together with initial condition

$$\pi_j = \text{Prob} (X(0) = j)$$

this will then completely define the continuous-time MC.

Note that entries in $A$ have dimensions $1/time$. 