03/30/06 Branching Processes

Remarks: The branching times do not necessarily have to be synchronized in real time for Galton-Watson model, but one is only viewing the process in terms of discrete generations.

\[ n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4 \]

\[ X_0 = 1 \quad X_1 = 2 \quad X_2 = 2 \quad X_3 = 4 \quad X_4 = 4 \]
If one wants to keep track of population size as a function of real time, this is possible with more sophisticated branching process models.

- Branching Processes in Biology, Kimivel and Axelrod

Applications (including more sophisticated variants)
- Gedry genealogical lines
- Populations with hindrances to reproduction, particularly if not too large
More applications
- proliferation of a virus (AIDS) which infects T-lymphocytes
- diversity of mutated samples in PCR
  - K. Mullis
- evolutionary studies
- nuclear fission
- queueing models
Mathematical Analysis
- general countable state MC
  tools not so efficient
  because prob. trans. matrix
  not easy to express
- exploit the structure of

the stochastic update rule
through probability
generating functions
(random sums)
\[ P(\mathbf{s}) = \langle \mathbf{X}_n \rangle = \sum_{j=0}^{\infty} \text{Prob}(\mathbf{X}_n = j) \mathbf{s}^j \]

Want to calculate this.

Given:

\[ P(\mathbf{s}) = \langle \mathbf{Y} \rangle \]

\[ \mathbf{Y} = -p + \text{for # offspring} \]

\[ P(\mathbf{s}) = \langle \mathbf{X}_0 \rangle \]

\[ \mathbf{X}_0 \]

Use stochastic update rule to get recursion formula for gen fn.
For example,

\[ P_1(s) = P_{1.0} \left( \sum_{i=1}^{k} \sum_{j=0}^{\infty} \sum_{s \in S} P(s) \right) \]

\[ P_{2,0}(s) = P_{2,0} \left( \sum_{i=1}^{k} \sum_{j=0}^{\infty} \sum_{s \in S} P(s) \right) \]

This lets us compute pdfs for \( S_n \) at any finite \( n \).

Simple question: What is the rate of growth of the expected population size?
\[ <X_n> = \left. \frac{d}{ds} P_{X_n}(s) \right|_{s=1} \]

\[ = \frac{d}{ds} \sum_{j=1}^{X_{n-1}} P_{X_j}(s) \bigg|_{s=1} \]

\[ = P'_{\sum}(1) \otimes P_{X_{n-1}}' \bigg( P_{\sum}(1) \bigg) \]

\[ = M \sum_{j=1}^{X_{n-1}} P'_{\sum}(1) \]

Avg offspring \[ = M <X_{n-1}> \]

So solving this recursion:

\[ <X_n> = M^n <X_0> \]

Can calculate variance, etc.

Similarly
One more fundamental question:
Probability of extinction
Or more generally, what happens in the long term?
Note that 0 is an absorbing state.

Therefore, if the states \{1, 2, \ldots, n\} form acommunic class, then they must all be transient.
(If it's possible at all to have 0 offspring)
— then only possibilities are extinction or \(X_n \to \infty\) (explosive growth)
By induction, composed with itself in times

\[
P^N(s) = \prod_{i=0}^{N-1} P(i, s)
\]

This holds because

\[
\prod_{i=1}^{N-1} x_i \leq \prod_{i=1}^{N} x_i
\]

with \(x_i \in \mathbb{N}_+\) and \(x_i > 0\)

Because

\[
\prod_{i=1}^{N} x_i = \prod_{i=1}^{N-1} x_i \cdot x_N
\]

\[
\prod_{i=1}^{N} x_i = \prod_{i=1}^{N} x_i \cdot x_N
\]

\[
\prod_{i=1}^{N} x_i = \prod_{i=1}^{N} x_i \cdot x_N
\]
Probability of extinction:
\[ a(k) = \text{Prob}(\bigcup_{n=1}^{\infty} \Xi_n = 0 \text{ for some } n > 0 \mid \Xi_0 = k) \]

Note \( a(k) = (a(1))^k, \ a(0) = 1 \)

Let \( a = a(1) \)
\[ a = \text{Prob}(\bigcup_{n=1}^{\infty} \Xi_n = 0 \mid \Xi_0 = 1) \]

First step analysis
\[ = \sum_{j=0}^{\infty} \text{Prob}(\bigcup_{n=1}^{\infty} \Xi_n = 0 \mid \Xi_j = j, \Xi_0 = 1) \times \text{Prob}(\Xi_j = j \mid \Xi_0 = 1) \]
\[ = \sum_{j=0}^{\infty} \text{Prob}(\bigcup_{n=1}^{\infty} \Xi_n = 0 \mid \Xi_j = j) \]
\[ \times \text{Prob}(\Xi_j = j \mid \Xi_0 = 1) \]
(Markov property)
\[
= \sum_{j=0}^{\infty} a(j) \ p_j
\]

\[
\rightarrow \text{time-homogeneity}
\]

\[
p_j = \text{Prob}(\sum = j)
\]

\[
a = \sum_{j=0}^{\infty} p_j a_j = \mathcal{P}_\Sigma(a)
\]

So the probability for extinction from 1 agent satisfies

\[
a = \mathcal{P}_\Sigma(a)
\]

But this can have multiple solutions—which is the right one?
B) \( X_n = X_0 \)

b) \( P_0 = \rho, \; p_i = 1 - \rho, \; P_j = 0 \) for \( j \geq 2 \)

\[ q = 1 \]

Non-trivial cases: \( p_j > 0 \) for some \( j \geq 2 \)

Study \( P^*_y(a) = \sum_{j=0}^{\infty} p_j a^j \)

\( 0 \leq P^*_y(0) \leq 1 \)

\( P^*_y(1) = 1 \)

\( P^*_{y'}(s) \geq 0 \) for \( 0 \leq s \leq 1 \)

\( P^*_{y''}(s) > 0 \) for \( 0 \leq s \leq 1 \)
Case I: \( P(x)^{\infty} > 1 \)

Case II: \( P(x)^{\infty} \leq 1 \)

Only solution: \( x = 1 \)
Which solution is correct in case I? The smallest solution to \( a \geq \frac{1}{\Sigma} \).

Proof:
Let \( q_N = \text{Prob}(\sum X = 0 | \sum Y = 1) \)
\[ = \frac{p_{X_N}(0)}{p_{X_N}(0) + p_{X_N}(0)} \]

Claim that \( q_N \leq \hat{a} \) for all \( N \)
where \( \hat{a} \) is smallest solution to \( a \geq \frac{1}{\Sigma} \).

Proof of claim by induction:
\( q_0 = 0 \leq \hat{a} \) \( \checkmark \)
\[ q_{N+1} = \sum Y \left( \frac{p_{X_N}(0)}{p_{X_N}(0)} \right) = \sum Y (a_N) \]

So \( a_n \leq \hat{a} \) \( \Rightarrow \)
\[ a_{N+1} \leq \frac{1}{\sum Y (\hat{a})} = \hat{a} \] \( \checkmark \)

(monotone increasing)
So now we know \( q_N \leq q \) for all \( N \).

\[
a = \lim_{N \to \infty} q_N \leq \lim_{N \to \infty} a = a
\]

1. \( a \approx \infty \) \( q > a \) \( \square \)
Summary of Long-Term Properties of Galton-Watson Process

1) If \( \mu \leq 1 \), and \( p_{ij} > 0 \) for some \( j > 2 \),
then \( \text{prob}(\text{extinction}) = 1 \).

2) Buring: If \( \mu = 1 \) and \( p_0 = 0, p_1 = 1 \),
then \( X_n \sim X_0 \)
- trivially recurrent

3) If \( \mu > 1 \), then
the probability for extinction starting
from \( k \) agents is \( \hat{\alpha}^k \) where
\( \hat{\alpha} \) is smallest root to
\( \hat{\alpha} = \frac{1}{p_{12}}(\hat{\alpha}), 0 \leq \hat{\alpha} < 1 \).

When not extinct, \( X_n \to \infty \)
with prob \( 1 - (1 - \hat{\alpha})^k \).