02/27/06 Discrete-Time Markov Chain Example

HW2 due Monday March 6 at 2 PM
Office hours on Friday moved to 2-3 PM.

1. Inspection Protocols for Manufacturing

Product ships if not inspected or if inspected and passed.

One possible inspection strategy:
Start by inspecting every product, but when you have inspected $M$ good products in a row, then switch to sampling only 1 in every $r$
products, (Regular interval). As soon as one inspects a defective product, return to sampling every product until you see $M$ good ones in a row again.

$M = 3 \quad r = 4$

Practical questions:

1) What fraction of shipped products are defective?

2) What fraction of products are inspected?
To answer these questions, we will use stochastic (Markov chain) model to account for uncertainy in which products are defective. Simplest model for defects in products: Each product has probability $p$ to be defective, independently of which other products are defective.

To be more realistic, to model clumps in defects, use on-off 2-state MC model. But we won't do this here.
Markov chain model for inspection:

We will define a Markov chain with state variable

\[ X_n = \# \text{ consecutive good products seen after } n \text{th inspection} \]
\[ n = \# \text{ inspection} \]
\[ S = \{0, \ldots, M\} \]

We don't need to include state of the product.

Stochastic update rule

\[ X_{n+1} = \begin{cases} 1 & \text{w/ prob } p(1-p) \\ 0 & \text{w/ prob } p \text{ if } X_n = 0 \\ X_n & \text{w/ prob } 1-p \text{ if } 1 \leq X_n \leq M-1 \end{cases} \]
Stochastic update rule:

\[ X_{n+1} = \begin{cases} 
0 & \text{w/ prob } 1-p \quad \text{if } X_n = 0 \\
X_n + 1 & \text{w/ prob } 1-p \quad \text{if } X_n > 0 \\
0 & \text{w/ prob } p \quad 1 \leq X_n \leq M-1 \\
M & \text{w/ prob } 1-p \\
0 & \text{w/ prob } p \quad \text{if } X_n = M
\end{cases} \]

Condense: \[ X_{n+1} = \begin{cases} 
\min(X_n + 1, M) & \text{w/ prob } 1-p \\
0 & \text{w/ prob } p
\end{cases} \]

Here the iid random variable input \( Z_n \) is \( Z_n = \begin{cases} 
0 & \text{w/ prob } 1-p \\
1 & \text{w/ prob } p
\end{cases} \)
Probability transition matrix:

\[ P_{ij} = \text{Prob}(X_{n+1} = j | X_n = i) \]

\[
P = \begin{pmatrix}
0 & 1-p & 2 & 3 & \cdots & \infty \\
1-p & 0 & 1 & 0 & \cdots & \infty \\
2 & 1-p & 0 & 1 & \cdots & \infty \\
3 & 2 & 1-p & 0 & \cdots & \infty \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\infty & \infty & \infty & \infty & \ddots & \ddots \\
\end{pmatrix}
\]

This MC is irreducible, aperiodic (think about reference states)

Then one has several pieces of info about long-time behavior.
There is a unique stationary distribution $\pi$

\begin{enumerate}
\item $\pi_j \geq 0$ for all $j \in S$
\item $\sum_{j \in S} \pi_j = 1$
\item $\pi \cdot p = \pi$
\end{enumerate}

and it is also a limit distribution.

Moreover, $\lim_{n \to \infty} \text{Prob}(X_n = j) = \pi_j$

Moreover, for any deterministic function $f$ on $S$

$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} f(X_n) = \sum_{j \in S} \pi_j f(j)$

Law of Large Numbers for Markov Chains.

with probability 1.

Resnick Sec. 2.12: proof.
Let's calculate $\pi$ to use in LLN.

Solve $\vec{\pi} \cdot P = \vec{\pi}$

0th equi

$\pi_0 \geq \pi_j \geq \pi_0$ for $j = 0, \ldots, M$

$1 \leq j \leq M-1$

$(1 - p) \pi_{j-1} = \pi_j$

$j > M$

$(1 - p) (\pi_{M-1} + \pi_M) = \pi_M$

(Con also use $\sum_{j=0}^{M} \pi_j = 1$ at will)

$\pi_0 = p$

$\pi_j = (1 - p)^{j-1} p$ for $1 \leq j \leq M-1$

$\pi_M = (1 - p)^M$ (after some algebra)
Consider first the fraction of products that are inspected.

\[ I = \frac{\text{fraction of items inspected in long run}}{\text{# products off the line during n inspections}} \]

\[ = \lim_{n \to \infty} \frac{\text{# inspections (n)}}{\text{# products off the line during n inspections}} \]

\[ T_n = \text{# products passed the came off the line between inspection \( n \) and \( n+1 \)} \]

\[ f(X_n) = \begin{cases} 1 & \text{if } 0 \leq X_n \leq M-1 \\ \frac{1}{n} & \text{if } X_n = M \end{cases} \]

\[ I = \lim_{n \to \infty} \frac{n}{\sum_{n'=1}^{n} T_{n'}} \]
\[= \lim_{n \to \infty} \left( \sum_{n'=1}^{n} \frac{T_{n'}}{n} \right)^{-1}\]

\[= \frac{1}{M} \sum_{j=0}^{M} f(j) \pi_j \text{ by LLN for M.C.}\]

where \(f(j) = \begin{cases} 1 & \text{for } 0 \leq j \leq M-1 \\ r & \text{for } j = M \end{cases}\)

\[= \frac{1}{M} \left( \sum_{j=0}^{M} \pi_j + r \pi_M \right) = \frac{1}{M} \left( \frac{\sum_{j=0}^{M} T_j + (r-1) T_M}{\sum_{j=0}^{M} T_j} \right)\]

\[I = \frac{1}{1 + (r-1)(1-p)^M} = \text{fraction of products inspected in long run.}\]
Second question: How good is the quality of shipped products?

Defect ratio

\[ \beta = \frac{\# \text{ defective shipped products}}{\# \text{ shipped products}} \]

\[ D_n = \# \text{ defective products shipped between inspection } \#n \text{ and } \#n+1 \]

\[ S_n = \# \text{ products shipped from (and including) inspection } \#n \]

up to \#n+1, but not including inspection \#n+1

\[ \beta = \lim_{N \to \infty} \frac{\sum_{n=1}^{N} D_n}{N} \leq \frac{\sum_{n=1}^{N} S_n}{N} \leq \frac{1}{N} \sum_{n=1}^{N} S_n \]