1. Find the general solution of the nonhomogeneous equation

\[ y'' - \frac{1}{t} y' + \frac{1}{t^2} y = -6t^2, \quad t > 0. \]  

(Hint: One solution of the homogeneous equation is \( y_1 = t \).)

Since we are given \( y_1(t) \), we can use reduction of order to solve this problem. We begin by assuming our solution will be of the form

\[ y(t) = v(t)y_1(t) = v(t)t. \]

Plugging this into (1), we get the following equation, and we can solve first for \( v'(t) \), then integrate to get \( v(t) \):

\[ (v''(t)t + 2v'(t)) - \frac{1}{t} (v'(t)t + v(t)) + \frac{1}{t^2} v(t)t = -6t^2 \]
\[ tv''(t) + v'(t) = -6t^2 \]
\[ \frac{d}{dt}(tv'(t)) = -6t^2 \]
\[ tv'(t) = \int (-6t^2) \, dt \]
\[ tv'(t) = -2t^3 + c_1 \]
\[ v'(t) = -2t^2 + c_1 t^{-1} \]
\[ v(t) = \int (-2t^2 + c_1 t^{-1}) \, dt \]
\[ = -\frac{2}{3} t^3 + c_1 \ln t + c_2. \]

So our solution is

\[ y(t) = v(t)y_1(t) \]
\[ = \left( -\frac{2}{3} t^3 + c_1 \ln t + c_2 \right) t \]
\[ = -\frac{2}{3} t^4 + c_1 t \ln t + c_2 t. \]

2. A mass weighing 3 lb stretches a spring 2 in. If the mass is pushed upward, contracting the spring a distance of 2 in, and then set in motion with a downward velocity of 2 ft/sec and if there is no damping, then find the position \( u(t) \) of the mass at any time \( t \). Determine the frequency, period and amplitude of the oscillation.

We start with the equation governing this system,

\[ mu'' + \gamma u' + ku = F(t). \]
We can calculate the parameters from the information given.

\[
\text{mass} = m = \frac{\text{weight}}{\text{gravity}} = \frac{3 \text{ lb}}{32 \text{ ft/s}^2} = \frac{3}{32} \text{ lb} \cdot \text{s}^2 = \frac{3}{32} \text{ slugs} \quad (15)
\]

\[
\text{damping constant} = \gamma = 0 \quad (16)
\]

\[
\text{spring constant} = k = \frac{\text{weight}}{\text{displacement}} = \frac{3 \text{ lb}}{1/6 \text{ ft}} = 18 \frac{\text{lb}}{\text{ft}} \quad (17)
\]

\[
\text{external force} = F(t) = 0. \quad (18)
\]

We are also given initial conditions \(u(0) = -2 \text{ in} \) and \(u'(0) = 2 \text{ ft/sec} \), so (14) becomes

\[
\frac{3}{32} u'' + 18u = 0, \quad u(0) = -\frac{1}{6} \text{ ft}, \quad u'(0) = 2 \text{ ft/sec} \quad (19)
\]

\[
u'' + 192u = 0. \quad (20)
\]

The characteristic equation is \(r^2 + 192 = 0\), with roots \(r = \pm 8\sqrt{3}i\). Thus our solution is

\[
u(t) = c_1 \cos \left(8\sqrt{3}t\right) + c_2 \sin \left(8\sqrt{3}t\right). \quad (21)
\]

We use the initial conditions to solve for \(c_1\) and \(c_2\).

\[
-\frac{1}{6} = u(0) = c_1 \quad (22)
\]

\[
2 = u'(0) = 8\sqrt{3}c_2 \quad \Rightarrow c_2 = \frac{1}{4\sqrt{3}}. \quad (23)
\]

So our equation for displacement of the spring is

\[
u(t) = -\frac{1}{6} \cos \left(8\sqrt{3}t\right) + \frac{1}{4\sqrt{3}} \sin \left(8\sqrt{3}t\right). \quad (24)
\]

To determine frequency, period and amplitude of the oscillation, we want to reformat our solution from

\[
u(t) = A \cos (\omega_0 t) + B \sin (\omega_0 t) \quad (25)
\]

to

\[
u(t) = R \cos (\omega_0 t + \delta), \quad (26)
\]

using the formulas

\[
A = R \cos \delta, \quad B = R \sin \delta \quad (27)
\]

which give us

\[
R = \sqrt{A^2 + B^2}, \quad \tan \delta = \frac{B}{A}. \quad (28)
\]

So in this case we have \(A = -1/6, B = 1/(4\sqrt{3})\), and the angular frequency of oscillation is \(\omega_0 = 8\sqrt{3} \text{ rad/sec}\). The period can be calculated from the angular frequency as

\[
T = \frac{2\pi}{\omega_0} = \frac{\pi}{4\sqrt{3}} \text{ sec}. \quad (29)
\]
Using (27), the amplitude is
\[ R = \sqrt{A^2 + B^2} = \sqrt{\left(\frac{-1}{6}\right)^2 + \left(\frac{1}{4\sqrt{3}}\right)^2} = \frac{\sqrt{7}}{12} \text{ ft.} \] (30)

We are not asked to calculate the phase \( \delta \).

3. A spring is stretched 5 cm by a force of 2 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 2 N when the velocity of the mass is 4 m/s. If the mass is pulled down 3 cm below its equilibrium position and given an initial upward velocity of 5 cm/s, determine the position \( u(t) \) of the mass at any time \( t \).

We start with the equation governing this system,
\[ mu'' + \gamma u' + ku = F(t). \] (31)

We can calculate the parameters from the information given.

mass = \( m \) = 2 kg (32)
damping constant = \( \gamma \) = \( \frac{\text{force}}{4 \text{ m/s}} \) = \( \frac{2 \text{ N}}{2 \text{ m}} = \frac{1 \text{ kg}}{2 \text{ s}} \) (33)

spring constant = \( k \) = \( \frac{\text{weight}}{\text{displacement}} \) = \( \frac{2 \text{ N}}{0.05 \text{ m}} = 40 \text{ N/m} = 40 \text{ kg/s}^2 \) (34)

external force = \( F(t) = 0 \text{ N} \). (35)

We can also determine initial conditions from the information. So (31) becomes
\[ 2u'' + \frac{1}{2}u' + 40u = 0, \quad u(0) = 0.03 \text{ m}, \quad u'(0) = -0.05 \text{ m/s}. \] (36)

The characteristic equation is \( 2r^2 + \frac{1}{2}r + 40 = 0 \), which has roots \( r = -1/8 \pm \sqrt{1279}/8i \).

The general solution is then
\[ u(t) = \exp \left[ -\frac{1}{8} t \right] \left[ c_1 \cos \left( \frac{\sqrt{1279}}{8} t \right) + c_2 \sin \left( \frac{\sqrt{1279}}{8} t \right) \right]. \] (37)

To find \( c_1 \) and \( c_2 \) we use the initial conditions.
\[ 0.03 = u(0) = c_1 \] (38)
\[ -0.05 = u'(0) = \frac{\sqrt{1279}}{8} c_2 - \frac{1}{8} c_1 \quad \Rightarrow \quad c_2 = -\frac{37}{100\sqrt{1279}} \approx -0.010346. \] (39)

Thus our solution is
\[ u(t) = \exp \left[ -\frac{1}{8} t \right] \left[ 0.03 \cos \left( \frac{\sqrt{1279}}{8} t \right) - \frac{37}{100\sqrt{1279}} \sin \left( \frac{\sqrt{1279}}{8} t \right) \right]. \] (40)

4. The displacement \( u(t) \) of a mass-spring-damper system is governed by the equation
\[ mu'' + \gamma u' + ku = F_0 \cos \omega t \] (41)
where \( m \) is the mass, \( \gamma \) is the damping constant, \( k \) is the spring constant, \( F_0 \) is the amplitude of the applied force, and \( \omega \) is the frequency of the periodic forcing.

(a) Consider first the undamped case with \( \gamma = 0 \). Find the value of \( \omega \) for which resonance occurs for the case \( m = 2, k = 10 \) and \( F_0 = 3 \).

Resonance occurs when the natural frequency \( \omega_0 \) equals \( \omega \), the frequency of the applied force. The natural frequency is given by \( \omega_0 = \sqrt{k/m} = \sqrt{5} \). So resonance occurs when \( \omega = \sqrt{5} \).

(b) Now consider a damped system with \( \gamma = 1 \). Determine the amplitude of the forced oscillation for the case \( m = 3, k = 2, F_0 = 1 \) and \( \omega = 1 \). (Hint: See page 208.)

Page 208 gives us the following formula for the amplitude of the forced oscillation:

\[
R = \frac{F_0}{\sqrt{m^2 (\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}, \quad \omega_0^2 = k/m. \tag{42}
\]

First calculate \( \omega_0^2 = k/m = 2/3 \). Then

\[
R = \frac{1}{\sqrt{3^2 (2/3 - 1)^2 + 1^2 \cdot 1^2}} = \frac{1}{\sqrt{2}}. \tag{43}
\]