Summary

Equations:

\[ \begin{align*}
    u' &= f(u, v) \\
    v' &= g(u, v)
\end{align*} \]

Steady-state:

\[ \begin{align*}
    f(u_0, v_0) &= 0 \\
    g(u_0, v_0) &= 0
\end{align*} \]

Stability: The Jacobian is

\[ J = \begin{pmatrix}
    \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\
    \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}
\end{pmatrix}. \]

The trace of \( J \) is

\[ \text{tr}(J) = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v}, \]

the determinant of \( J \) is

\[ \det(J) = \frac{\partial f}{\partial u} \frac{\partial g}{\partial v} - \frac{\partial f}{\partial v} \frac{\partial g}{\partial u}, \]

and the eigenvalues of \( J \) are

\[ r_\pm = \frac{1}{2} \left( \text{tr}(J) \pm \sqrt{[\text{tr}(J)]^2 - 4 \det(J)} \right). \]

In the theorems below, \( J \) is evaluated at \((u_0, v_0)\).

**Theorem (version 1):** The steady-state is asymptotically stable if the eigenvalues \( r_1, r_2 \) of \( J \) satisfy \( \text{Re}(r_1) < 0 \) and \( \text{Re}(r_2) < 0 \). The steady-state is unstable if \( \text{Re}(r_1) > 0 \) or \( \text{Re}(r_2) > 0 \).

**Theorem (version 2):** The steady-state is asymptotically stable if \( \text{tr}(J) < 0 \) and \( \det(J) > 0 \). It is unstable if \( \det(J) < 0 \) or \( \text{tr}(J) > 0 \).

**Hamiltonian:** For \( mu'' = F(u) \),

\[ H = \frac{1}{2} mu_t^2 + V(u) \]

where \( V(u) = -\int F(u)du \).
Steady-States and Stability

The general version of the problem considered is
\[
\frac{d}{dt} y = f(y) \tag{1}
\]

Examples

1. Pendulum
   The differential equation is
   \[
   \theta'' + \omega^2 \sin(\theta) = 0,
   \]
   where \(\omega^2 = g/\ell\). Letting \(v = \theta'\) this can be written in system form as
   \[
   \theta' = v
   \]
   \[
   v' = -\omega^2 \sin \theta. \quad \blacksquare
   \]

2. Michaelis-Menten
   The system is
   \[
   S' = -k_1 ES + k_{-1}(K_0 - E)
   \]
   \[
   E' = -k_1 ES + (k_2 + k_{-1})(K_0 - E)
   \]
   
   This can be written in vector form as in [1], where
   \[
   y = \begin{pmatrix} S \\ E \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} -k_1 ES + k_{-1}(K_0 - E) \\ -k_1 ES + (k_2 + k_{-1})(K_0 - E) \end{pmatrix}
   \]
   
   Note that \(k_1, k_{-1}, k_{-2}, \) and \(K_0\) are positive constants. \(\blacksquare\)

Steady-State Solution(s)

For \(y' = f(y)\), a steady-state solution \(y_0\) is a constant vector that satisfies \(f(y_0) = 0\).

Examples

1. Pendulum
   For \((\theta_0, v_0)\) to be a steady-state it is required that
   
   \[
   v_0 = 0
   \]
   \[
   -\omega^2 \sin \theta_0 = 0
   \]
   
   There are two distinct steady-states and they are \(\theta_0 = v_0 = 0\) and \(\theta_0 = \pi, v_0 = 0\). \(\blacksquare\)
2. Michaelis-Menten

For \((S_0, E_0)\) to be a steady-state it is required that

\[-k_1E_0S_0 + k_{-1}(K_0 - E_0) = 0\]
\[-k_1E_0S_0 + (k_2 + k_{-1})(K_0 - E_0) = 0\]

From this one finds that \(E_0 = K_0\) and \(S_0 = 0\).

Stability

The differential equations are

\[
\begin{align*}
  u' &= f(u, v) \\
  v' &= g(u, v).
\end{align*}
\]

Assume that \((u_0, v_0)\) is a steady-state, which means that \(u_0\) and \(v_0\) are constants that satisfy

\[
\begin{align*}
  f(u_0, v_0) &= 0 \\
  g(u_0, v_0) &= 0.
\end{align*}
\]

The reason for considering stability comes from the question: If we start the solution near \((u_0, v_0)\), what happens? To put this in more mathematical terms, we are assuming that the initial conditions are

\[
\begin{align*}
  u(0) &= u_0 + \varepsilon_1 \\
  v(0) &= v_0 + \varepsilon_2,
\end{align*}
\]

where \(\varepsilon_1\) and \(\varepsilon_2\) are tiny numbers (not both nonzero). To determine what happens, we will approximate \(f(u, v)\) and \(g(u, v)\) for \((u, v)\) near \((u_0, v_0)\) using Taylor’s theorem. Using this theorem, we get that

\[
\begin{align*}
  f(u, v) &= f(u_0, v_0) + (u - u_0)f_u(u_0, v_0) + (v - v_0)f_v(u_0, v_0) \\ 
  &+ \frac{1}{2}(u - u_0)^2 f_{uu} + (u - u_0)(v - v_0)f_{uv} + \frac{1}{2}(v - v_0)^2 f_{vv} + \cdots,
\end{align*}
\]

\[
\begin{align*}
  g(u, v) &= g(u_0, v_0) + (u - u_0)g_u(u_0, v_0) + (v - v_0)g_v(u_0, v_0) + \cdots \\ 
  &+ \frac{1}{2}(u - u_0)^2 g_{uu} + (u - u_0)(v - v_0)g_{uv} + \frac{1}{2}(v - v_0)^2 g_{vv} + \cdots.
\end{align*}
\]

Note that the right hand side in both of the above formulas consists of the sum of a constant, linear terms (these involve first derivatives), quadratic terms (these involve second derivatives), etc. We are going to include everything through the linear terms. Since \(f(u_0, v_0) = 0\) and \(g(u_0, v_0) = 0\), the resulting approximations are

\[
\begin{align*}
  f(u, v) &\approx (u - u_0)f_u(u_0, v_0) + (v - v_0)f_v(u_0, v_0) \\
  g(u, v) &\approx (u - u_0)g_u(u_0, v_0) + (v - v_0)g_v(u_0, v_0).
\end{align*}
\]
With this, the reduced (approximate) problem to solve is

\[ u' = (u - u_0)f_u(u_0, v_0) + (v - v_0)f_v(u_0, v_0) \]
\[ v' = (u - u_0)g_u(u_0, v_0) + (v - v_0)g_v(u_0, v_0). \]

This can be written in vector form as

\[ u' = J(u - u_0), \tag{2} \]

where

\[ u = \begin{pmatrix} u \\ v \end{pmatrix}, \quad u_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \]

and

\[ J = \begin{pmatrix} f_u(u_0, v_0) & f_v(u_0, v_0) \\ g_u(u_0, v_0) & g_v(u_0, v_0) \end{pmatrix}. \]

The matrix \( J \) is known as the Jacobian matrix of \( f \) evaluated at \( u_0 \). The eigenvalues of this matrix are

\[ r = \frac{1}{2} \left( f_u + g_v \pm \sqrt{(f_u + g_v)^2 - 4(f_u g_v - f_v g_u)} \right) \]
\[ = \frac{1}{2} \left( \text{tr}(J) \pm \sqrt{\text{tr}(J)^2 - 4 \det(J)} \right), \tag{3} \]

where \( \text{tr}(J) = f_u + g_v \). The general solution of (2) is

\[ u = u_0 + c_1 x_1 e^{r_1 t} + c_2 x_2 e^{r_2 t}, \]

where \( r_1 \) and \( r_2 \) are the eigenvalues of \( J \), and \( x_1 \) and \( x_2 \) are the respective eigenvectors. This assumes that \( J \) is not defective. The constants \( c_1 \) and \( c_2 \) are determined from the initial conditions.

Conclusions:

1. If \( \text{Re}(r_1) < 0 \) and \( \text{Re}(r_2) < 0 \), then \( u \to u_0 \) as \( t \to \infty \) (no matter what \( c_1 \) and \( c_2 \) are): in this case we say \( u_0 \) is asymptotically stable.

2. If \( \text{Re}(r_1) > 0 \) or \( \text{Re}(r_2) > 0 \), then \( u \) can grow exponentially: in this case we say \( u_0 \) is unstable.

3. In the remaining cases our method fails (i.e. the steady-state could be stable or unstable or possibly have some other property).

Labor saving observation: The formula in (3) can be used to derive a simple test for stability. First, note that if \( \det(J) < 0 \) then the + eigenvalue is a positive number. If \( \det(J) > 0 \), and \( |\text{tr}(J)|^2 - 4 \det(J) \geq 0 \), then the + eigenvalue is positive if \( \text{tr}(J) > 0 \). Finally, \( \det(J) > 0 \), and \( |\text{tr}(J)|^2 - 4 \det(J) < 0 \), then the + eigenvalue is complex-valued with positive real part if \( \text{tr}(J) > 0 \). Based on this, the above conclusions can be rewritten as:

1. If \( \text{tr}(J) < 0 \) and \( \det(J) > 0 \), then \( u_0 \) is asymptotically stable.

2. If \( \text{tr}(J) > 0 \) or \( \det(J) < 0 \), then \( u_0 \) is unstable.

3. In the remaining cases our method fails (i.e. the steady-state could be stable or unstable or possibly have some other property).
Examples

1. Pendulum

As found earlier, there are two distinct steady-states and they are \( \theta_0 = v_0 = 0 \) and \( \theta_0 = \pi, \ v_0 = 0 \). Since \( f(\theta,v) = v \) and \( g(\theta,v) = -\omega^2 \sin \theta \), the Jacobian is
\[
J = \begin{pmatrix}
f_\theta & f_v \\
g_\theta & g_v
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-\omega^2 \cos \theta_0 & 0
\end{pmatrix}.
\]

Consequently, \( \text{tr}(J) = 0 \) and \( \det(J) = \omega^2 \cos \theta_0 \).

(a) \( \theta_0 = \pi, \ v_0 = 0 \): Since \( \det(J) = -\omega^2 < 0 \) it follows that this steady-state is unstable.

(b) \( \theta_0 = 0, \ v_0 = 0 \): Since \( \text{tr}(J) = 0 \) and \( \det(J) > 0 \) we are not able to make a conclusion.

2. Michaelis-Menten

The Jacobian is (before evaluating it at \( (S_0, E_0) \))
\[
J = \begin{pmatrix}
f_S & f_E \\
g_S & g_E
\end{pmatrix}
= \begin{pmatrix}
-k_1 E & -k_1 S - k_{-1} \\
-k_1 E & -k_1 S - (k_2 + k_{-1})
\end{pmatrix}.
\]

Evaluating this at the steady state \( (S_0, E_0) = (0, K_0) \),
\[
J = \begin{pmatrix}
-k_1 K_0 & -k_{-1} \\
-k_1 K_0 & -(k_2 + k_{-1})
\end{pmatrix}.
\]

Since \( \text{tr}(J) = -k_1 K - (k_2 + k_{-1}) < 0 \) and \( \det(J) = k_1 k_2 K_0 > 0 \), then the steady state is asymptotically stable.

Figure 1: Michaelis-Menten: all paths lead to the steady-state \( k_{-1} = k_{-1} = k_2 = K_0 = E_0 = 1 \).
3. Problem 8, Sec 9.3

\[ x' = x - x^2 - xy \]
\[ y' = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy \]

*Steady-states* (derived in class): \((0, 0), (0, 2), (1, 0), (1/2, 1/2)\)

*Stability* (derived in class):

\[ J = \begin{pmatrix}
1 - 2x - y & -x \\
-\frac{3}{4}y & \frac{1}{2} - \frac{1}{2}y - \frac{3}{4}x
\end{pmatrix} \]

\((0, 0)\):

\[ J = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{2}
\end{pmatrix} \]

Since \(\text{tr}(J) = 3/2 > 0\), this point is unstable

\((0, 2)\): asymptotically stable

\((1, 0)\): asymptotically stable

\((1/2, 1/2)\): unstable

*Plots and Observations*

![Figure 2: Phase plane plot. Open circle: asymptotically stable point. Square: unstable point.](image)

Observation: In the above plot, the four solid dots are the starting points and they are placed near the the asymptotically stable steady-state \((1, 0)\). In each case the resulting solution approaches \((1, 0)\) as \(t \to \infty\) (which is expected).
Observation: The values for $x(t)$ and $y(t)$ are graphed as a function of time for the purple/magenta curve shown in the previous phase plane plot. It shows that as $t \to \infty$, $x \to 1$ and $y \to 0$. This is the same conclusion made earlier, which is that $(x, y) \to (1, 0)$ as $t \to \infty$.

Observation: Using four different starting points that are not particularly close to an asymptotically stable steady-state, the solution is unpredictable. In three cases it does approach a steady-state, although that point is not necessarily the one closest to the starting point. In the fourth case (the red curve) the solution does not approach any steady-state and appears to simply become unbounded. Note the green curve makes a large loop (something similar to what the purple/magenta curve does), and ends up approaching $(0, 2)$. ■
4. Predator-Prey (Sec 9.5)

\[ u' = au - buv \]
\[ v' = -cv + duv \]

Steady-states (derived in class): (0, 0), (c/d, a/b) (note that a, b, c and d are positive)

Stability (derived in class):

\[ J = \begin{pmatrix} a - bv & -bu \\ dv & -c + du \end{pmatrix} \]

(0, 0):

\[ J = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix} \]

Since \( \text{det}(J) = -ac < 0 \), this point is unstable

(c/d, a/b): indeterminate

Plots and Observations

Observations: In the upper plot, the four solid dots are the starting points. In each case the solution curve encircles the steady-state point (c/d, a/b) indicated with the x. In the lower plot, the values for u(t) and v(t) are graphed as a function of time for the purple/magenta curve in the upper plot. Note that both functions appear to be periodic. Also, in the calculation, \( a = b = c = d = 1 \).
Periodic Solutions, Hamiltonians, and Conservation of Energy

The question considered is this: can you determine if the solution is periodic? What will be seen is that for many problems it is possible to determine this without actually solving the problem.

Example: mass-spring equation

The equation for a mass and spring is

\[ mu'' + ku = 0. \] \hspace{1cm} (4)

The general solution of this is \( u = R \cos(\omega t - \varphi) \), where \( \omega = \sqrt{k/m} \). This is periodic, and our goal is to see if we can determine this without actually solving the problem.

Multiplying (4) by the velocity \( u' \), it is possible to rewrite the equation as

\[ \frac{d}{dt} \left( \frac{1}{2} mu_t^2 + \frac{1}{2} ku^2 \right) = 0. \] \hspace{1cm} (5)

This is the reason for introducing the function

\[ H(u, u_t) = \frac{1}{2} mu_t^2 + \frac{1}{2} ku^2. \] \hspace{1cm} (6)

Physically \( H \) is the sum of the kinetic and potential energy of the system and it’s called a Hamiltonian. What (5) shows is that \( H \) does not change in time, which means that the total energy is conserved.

Assuming the initial conditions are \( u(0) = u_0 \) and \( u'(0) = 0 \), then integrating (5) we have that

\[ \frac{1}{2} mu_t^2 + \frac{1}{2} ku^2 = \frac{1}{2} ku_0^2. \]

This can be rewritten as

\[ u^2 + (m/k)v^2 = u_0^2, \] \hspace{1cm} (7)

where \( v = u' \) is the velocity.

Particular case

Taking \( m = 1 \) and \( k = 4 \), then (7) becomes

\[ u^2 + \frac{1}{4}v^2 = u_0^2. \]

This is an equation for an ellipse and it is illustrated in Figure 4. Because of this we can make two significant observations about the solution:

1. No matter what values \( u \) and \( v \) might have, the solution \( (u,v) \) will always be located somewhere on this ellipse.
2. The solution never comes to rest or changes directions on this curve. This needs to be verified. This is done by noting that the solution can only change direction, or stop altogether, at points where \( v = 0 \). There are two points on the ellipse where this happens: \((u_0, 0)\) and \((-u_0, 0)\). Also, note that from the differential equation (4), \( v' = -(k/m)u \). The explanation of why the solution never comes to rest or changes directions is:

<table>
<thead>
<tr>
<th>At ((u_0, 0)), ( v' = -(k/m)u_0 &lt; 0 ), which means that ( v ) is decreasing at this point. Consequently the solution is moving downward along the curve (and is not stopping).</th>
</tr>
</thead>
<tbody>
<tr>
<td>At ((-u_0, 0)), ( v' = (k/m)u_0 &gt; 0 ), which means that ( v ) is increasing. Consequently the solution is moving upward along the curve (and is not stopping).</td>
</tr>
</tbody>
</table>

The conclusion we make from these observations is that the solution is periodic, and simply keeps going around and around the ellipse in a clock-wise manner.

**General Case: Conservation of Energy**

Similar to the mass-spring example, it’s assumed that the equation to solve is

\[
mu'' = F(u),
\]

where \( F(u) \) is the forcing function (it can depend on \( u \) but not \( u' \)). To find the Hamiltonian, multiple by the velocity \( u' \) to obtain

\[
mu'u'' = F(u)u'.
\]  

(8)
The next step requires two observations. The first one is

\[ \frac{d}{dt} \left( \frac{1}{2} \mu u^2_t \right) = \mu u'' \].

We want to do the same thing for the forcing term, which means we want to find a function \( V(u) \) so that

\[ \frac{d}{dt} V(u) = -F(u)u'. \]

Using the chain rule, the above equation can be written as \( V'(u)u' = -F(u)u' \). So, we want \( V'(u) = -F(u) \), and from this we obtain the equation

\[ V(u) = -\int F(u)du. \]

For example, if \( F(u) = -ku \), then \( V(u) = \frac{1}{2}ku^2 \), or if \( F(u) = e^{3u} - u^3 \), then \( V(u) = -\frac{1}{3}e^{3u} + \frac{1}{4}u^4 \).

Assuming we have found \( V(u) \), then (8) becomes

\[ \frac{d}{dt} \left( \frac{1}{2} \mu u^2_t + V(u) \right) = 0. \]

Setting

\[ H = \frac{1}{2} \mu u^2_t + V(u), \quad (9) \]

then the conclusion is that \( H = \text{constant} \). The function \( H \) is called a Hamiltonian, and it is the sum of the kinetic energy \( \frac{1}{2} \mu u^2_t \) and the potential energy \( V(u) \). The conclusion is nothing more than the statement that the total energy is constant.

**Example: pendulum**

The equation is

\[ \theta'' + \omega^2 \sin \theta = 0. \]

The Hamiltonian will be derived two ways:

1. **Version 1 (Direct Approach):** Multiplying the equation by the angular velocity \( \theta' \), the equation can be written as

\[ \frac{d}{dt} \left( \frac{1}{2} \theta^2 t - \omega^2 \cos \theta \right) = 0. \quad (10) \]

The Hamiltonian in this case is

\[ H(\theta, v) = \frac{1}{2} v^2 - \omega^2 \cos \theta, \quad (11) \]

where \( v = \theta' \).

2. **Version 2 (Using Formulas):** The kinetic energy is \( \frac{1}{2} \theta^2_t \). As for the potential energy, since \( F(\theta) = -\omega^2 \sin \theta \), then \( V(\theta) = -\int F(\theta)d\theta = -\omega^2 \cos \theta \). Combining these energies, one gets (11).
For the mass-spring example the equation $H = constant$ gave rise to the ellipse in (7) and Figure 4. For the pendulum, the equation $H = constant$ is more complicated. The easiest way to determine what these curves look like is to plot the surface $z = H(\theta, v)$ and then construct the level curves for the surface (MATLAB has a built in command to do this). The result is shown in Figure 5 (next page).

We will consider the case of when the initial conditions are $\theta(0) = \theta_0$, $v(0) = 0$, where $0 < \theta_0 < \pi$. This means that the curves we are considering are the three ellipse-like curves that encircle the point $x$ in Figure 5. A generic version of this is shown in the figure below.

As before, we can make two significant observations about the solution:

1. No matter what values $\theta$ and $v$ might have, the solution $(\theta, v)$ will always be located somewhere on this curve.

2. The solution never comes to rest or changes directions on this curve. To verify this we need to check the $v = 0$ points on the curve. First, note that from the differential equation, we find that $v' = -\omega^2 \sin \theta$. Consequently, at $(\theta_0, 0)$, $v' < 0$, which means that $v$ is decreasing at this point and so the solution is moving downward along the curve (and is not stopping). Similarly, at $(-\theta_0, 0)$, $v' > 0$, and so the solution is moving upward along the curve (and is not stopping).

The conclusion we make from these observations is that the solution is periodic, and simply keeps going around and around the curve in a clock-wise manner. To verify this, the solution computed using MATLAB is shown in Figure 6.

Parting Comments

The method used to find a Hamiltonian works mostly on problems originating from equations of the form $ma = F$. For equations not originating from a force balance law, finding a Hamiltonian can be challenging. As an example, the predator-prey problem has a periodic solution that is similar to the periodic solution for the pendulum problem, but finding something that is effectively a Hamiltonian is not so easy. If you are interested in what it is, look at the Scholarpedia page for predator-prey, which can be found here (on that page the equations are referred to by their other name, which is the Lotka-Volterra model).
Figure 5: Energy surface $H(\theta, v)$ and its associated level curves for the pendulum. Open squares: unstable steady-states. x: indeterminate steady-state.
Figure 6: Top: Phase plane plot for pendulum when $g/\ell = 4$. The four solid dots are the starting points. In each case the solution curve encircles the steady-state point $(0, 0)$ indicated with the x. Bottom: The values for $\theta(t)$ and $v(t)$ are graphed as a function of time for the purple/magenta curve in the upper plot. Note that both functions appear to be periodic.