Summary: Second Order, Linear DEs

1. Homogeneous

Theory

If \( y_1(t) \) and \( y_2(t) \) are linearly independent solutions of
\[
y'' + p(t)y' + q(t)y = 0,
\]
then the general solution can be written as \( y = c_1 y_1 + c_2 y_2 \), where \( c_1 \) and \( c_2 \) are constants.

General Solution

To solve \( ay'' + by' + cy = 0 \), where \( a \), \( b \) and \( c \) are constants with \( a \neq 0 \), assume \( y = e^{rt} \). This leads to the solving \( ar^2 + br + c = 0 \), and from this the resulting general solution is:

Two Real Roots: \( r = r_1, r_2 \) (with \( r_1 \neq r_2 \))
\[
y = c_1 e^{r_1 t} + c_2 e^{r_2 t}
\]
Complex Roots: \( r = \lambda \pm i\mu \) (with \( \mu \neq 0 \))
\[
y = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t) \quad \text{where } c_1, c_2 \text{ are real-valued}
\]
\[
\bigtriangledown y = Re^{\lambda t} \cos(\mu t - \varphi) \quad \text{where } R \geq 0, \text{ and } 0 \leq \varphi < 2\pi
\]
One Real Root: \( r = \lambda \)
\[
y = c_1 e^{\lambda t} + c_2 t e^{\lambda t}
\]

Free Vibrations

Undamped: \( my'' + ky = 0 \) with \( k \) and \( m \) positive
\[
y = R \cos(\omega_0 t - \varphi) \quad \text{where } \omega_0 = \sqrt{k/m} \text{ is called the natural frequency}
\]
Damped: \( my'' + \gamma y' + ky = 0 \) with \( k, \gamma, \) and \( m \) positive

Over-damped: two real roots \( (\gamma^2 > 4km) \)
Critically damped: one real root \( (\gamma^2 = 4km) \)
Under-damped: two complex roots \( (\gamma^2 < 4km) \)
2. Inhomogeneous Theory

The general solution of

\[ y'' + p(t)y' + q(t)y = g(t), \]

can be written as

\[ y(t) = y_p(t) + y_h(t), \]

where \( y_p \) is a particular solution and \( y_h \) is a general solution of the associated homogeneous DE.

Particular Solution

Undetermined Coefficients (requires \( p \) and \( q \) to be constants)

\[
\begin{array}{|c|c|}
\hline
\text{\( g(t) \) contains} & \text{\( y_p(t) \) contains} \\
\hline
e^{at} & e^{at} \\
\cos(\omega t) \text{ or } \sin(\omega t) & \cos(\omega t), \sin(\omega t) \\
t^n & t^n, \ t^{n-1}, \ldots, \ 1 \\
t^n e^{at} & t^n e^{at}, \ t^{n-1} e^{at}, \ldots, \ e^{at} \\
e^{at} \cos(\omega t) \text{ or } e^{at} \sin(\omega t) & e^{at} \cos(\omega t), \ e^{at} \sin(\omega t) \\
\hline
\end{array}
\]

The table above contains examples of first guesses for \( y_p \). Note that \( n \) must be a non-negative integer. Also, if one of the functions in the left column is a solution of the associated homogeneous DE, then the functions in the right column need to be multiplied by \( t \) or \( t^2 \).

Wronskian Version (Variation of Parameters)

\[
y_p(t) = -y_1(t) \int_t^\tau \frac{y_2(s)g(s)}{W(y_1, y_2)} \, ds + y_2(t) \int_t^\tau \frac{y_1(s)g(s)}{W(y_1, y_2)} \, ds
\]