Solutions to Midterm

(a) Due to (Pb), it is:

\[
\begin{align*}
\text{max} & \quad b^Ty - u^Tz \\
\text{s.t.} & \quad A^Tz - z = c \\
& \quad z \geq 0
\end{align*}
\]

Take \( y = 0 \), \( z_i = \max \{ 0, -c_i \} \), \( i = 1, \ldots, n \).
This is feasible.

(b) Consider the primal-dual pair of LPs:

\[
\begin{align*}
\text{min} & \quad -x_i \\
\text{s.t.} & \quad Ax = b \quad (P_i) \\
& \quad x \geq 0 \\
& \quad \text{max} \quad b^Ty_i \\
& \quad \text{s.t.} \quad A^T y_i \leq -e_i \quad (D_i)
\end{align*}
\]

where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T \), the \( i \)-th unit vector.

If \( x_i \) is bounded then \((P_i)\) has finite optimal value, so \((D_i)\) has finite optimal value and is feasible.

Let \( y_i \) be a feasible solution to \((P_i)\).

Let \( \hat{y}_i \) be a feasible solution to \((D)\).

Then \( y_i + \alpha \hat{y}_i \) is feasible in \((D)\) for any \( \alpha \geq 0 \), since

\[
A^T(y_i + \alpha \hat{y}_i) \leq A^T y_i + \alpha A^T \hat{y}_i \leq b - \alpha e_i \leq b
\]

Also, let \( s_\alpha = b - A^T(y_i + \alpha \hat{y}_i) \geq \alpha e_i \).

As \( \alpha \to \infty \), \( s_\alpha \to \infty \). So \( s_\alpha \) is unbounded. \( \Box \).
(a) $BB^T = LQ\bar{Q}^TL^T = LL^T$

(b) $B^Ty = c \Rightarrow Q^TL^Ty = c$  
    
    $L^Ty = \bar{Q}c$

    So:
    (i) Calculate $\bar{Q}c$
    (ii) Backsolve for $y$.

    $Bd = a \Rightarrow L\bar{Q}d = a$

    So:
    (i) Forward solve for $u = \bar{Q}d$
    (ii) Calculate $d = \bar{Q}^Tu$

(c) $B^Ty = c \Rightarrow BB^Ty = Bc$  
    
    $LL^Ty = Bc$

    So:
    (i) Calculate $Bc$
    (ii) Forward substitude to find $u$ satisfying $Lu = Bc$
    (iii) Back substitute to find $y$.

    $Bd = a \Rightarrow BB^Tp = a$ with $B^tp = d$
    
    $LL^Tp = a$

    So:
    (i) Find $p$ using forward & back substitution
    (ii) Calculate $d = B^tp$.

(d) From the hint, we can assume there exist $\bar{L}, \bar{Q}$ with $B = \bar{L}\bar{Q}$.

Then $LL^T = \bar{L}\bar{Q}\bar{Q}^T\bar{L}^T$ since $\bar{Q}$ orthogonal

$= \bar{B}\bar{B}^T$

$= (B + (a-b)e_p^T)(B + (a-b)e_p^T)^T = (B + (a-b)e_p^T)(B + e_p(a-b)^T)$

$= BB^T + (a-b)b^T + b(a-b)^T + (a-b)(c-b)^T$ since $Be_p = b$ and $e_p^Te_p = 1$

$= LL^T + aa^T - bb^T$ since $BB^T = LL^T$. 

Assume \( \text{int } R_{\varepsilon} \neq \emptyset \).

Let \( x_{\varepsilon} \in \text{int } R_{\varepsilon} \) satisfy \( e^T x_{\varepsilon} = R_{\varepsilon} \).

Let \( x^* \) solve (P).

Define \( x_\varepsilon = x^* + \frac{\varepsilon}{\varepsilon'} (x_{\varepsilon'} - x^*) \quad (1) \)

Then \( e^T x_\varepsilon = e^T x^* + \frac{\varepsilon}{\varepsilon'} e^T (x_{\varepsilon'} - x^*) \)

\[
= (1 - \frac{\varepsilon}{\varepsilon'}) e^T x^* + \frac{\varepsilon}{\varepsilon'} e^T x_{\varepsilon'} \leq (1 - \frac{\varepsilon}{\varepsilon'}) e^T x^* + \frac{\varepsilon}{\varepsilon'} (e^T x^* + \varepsilon')
\]

\[
= e^T x^* + \varepsilon 
\]

So, \( x_\varepsilon \in R_\varepsilon \), so \( R_{\varepsilon} \geq e^T x_\varepsilon \).

Also, \( x_{\varepsilon} \) is a cone from \( 0 \), we have

\[
e^T x_\varepsilon = e^T x^* + \frac{\varepsilon}{\varepsilon'} e^T x_{\varepsilon'} - \frac{\varepsilon}{\varepsilon'} e^T x^*
\]

So, \( R_{\varepsilon'} = e^T x_{\varepsilon'} = \frac{\varepsilon'}{\varepsilon^2} e^T x_{\varepsilon} + (1 - \frac{\varepsilon'}{\varepsilon^2}) e^T x^* \leq \frac{\varepsilon'}{\varepsilon^2} R_{\varepsilon} \) since \( e^T x_{\varepsilon} \leq R_{\varepsilon}, e^T x^* \geq 0, 1 - \frac{\varepsilon'}{\varepsilon^2} \leq 0.\)
Eq:

Supply = 5  Demand = 5  Supply = 10  Demand = 10

\[ \begin{align*}
0 & \rightarrow 2 & \text{Cost} = 1 \\
2 & \rightarrow 3 & \text{Cost} = 100 \\
3 & \rightarrow 4 & \text{Cost} = 1
\end{align*} \]

Optimal soln: \( x_{12} = 5, \ x_{34} = 10, \ x_{23} = 0 \),
\[ \text{Cost} = 15. \]

Reduce demand at node 2 to 4.
Reduce supply at node 3 to 9.

Optimal soln: \( x_{12} = 5, \ x_{23} = 1, \ x_{34} = 10 \)
\[ \text{Cost} = 115. \]
Flow are shown.
Value: -11.

Dual constraints: \( \pi_j - \pi_i - y_{ij} \leq c_{ij} \) for each arc \((i, j)\).

\( y_{ij} \) are dual multipliers for upper bound constraints and must be nonnegative.

\( \pi_i \) are free.

For basic variables, \( y_{ij} = 0 \) and \( \pi_j - \pi_i = c_{ij} \).

So:
\[
\begin{align*}
\pi_2 - \pi_1 &= 0 \\
\pi_4 - \pi_3 &= 0 \\
\pi_1 - \pi_4 &= -10
\end{align*}
\]

Take \( \pi_1 = 0 \). Then \( \pi_2 = 0, \pi_3 = 0, \pi_4 = 10 \).

For nonbasic variables:
- \((1, 3)\): at lower bound, so \( y_{13} = 0 \).
  Reduced cost: \( c_{13} + \pi_1 \bar{y}_{13} = -12 \).
- \((2, 3)\): at upper bound, so \( c_{ij} = \pi_j - \pi_i - y_{ij} \)
  So: \( y_{23} = \pi_3 - \pi_2 - c_{23} = +11 \).
- \((2, 4)\): at lower bound, so \( y_{24} = 0 \)
  Reduced cost: \( c_{24} + \pi_2 - \pi_4 = -12 \).

So, either \( x_{13} \) or \( x_{24} \) enter the basis.
Arbitrarily choose to bring $x_{13}$ into basis:

Flow conservation requires this choice of flow.

Largest $t$ before capacity constraint violated is $t = 0$. $x_{34}$ leaves basis.

New bfs: $x_{12} = 1, x_{13} = 0, x_{14} = 1$

New bfs: $x_{23} = 1, x_{34} = 1, x_{24} = 0$.

Find $\pi_c$ from basic variables:

$$\pi_c = \pi_1 = 0,$$
$$\pi_2 = \pi_1 - 2,$$
$$\pi_3 = \pi_1 - 10.$$

Take $\pi_1 = 0$. Then $\pi_2 = 0, \pi_3 = -2, \pi_4 = 10$.

For non-basic variables:

$(2,3)$: At upper bound, so $c_{ij} = \pi_j - \pi_i - y_{ij}$.

So $y_{23} = \pi_2 - \pi_1 - c_{23} = -1$.

$(2,4)$: At lower bound, so $y_{24} = 0$.

Reduced cost: $c_{24} + \pi_2 - \pi_4 = -12$.

$(3,4)$: At upper bound, so $c_{ij} = \pi_j - \pi_i - y_{ij}$.

So $y_{34} = \pi_4 - \pi_3 - c_{34} = 12$. 
So choose to drop $x_{24}$ into the basis:

\[ 1+ \kappa \]

\[ 1+ \kappa \rightarrow 2 - \kappa \rightarrow 4 \rightarrow 3 \]

$x_{12}$ leaves basis, another degenerate pivot.

New b.b.: basic: $x_{13} = 0$, $x_{24} = 0$, $x_{41} = 1$.

New b.b.: $x_{12} = 1$, $x_{23} = 1$, $x_{34} = 1$

Find $\pi_i$ from basis variables:

\[
\begin{align*}
\pi_3 - \pi_4 &= -2 \\
\pi_1 - \pi_4 &= -10 \\
\pi_4 - \pi_2 &= -2
\end{align*}
\]

Take $\pi_1 = 0$. Then $\pi_3 = -2$, $\pi_4 = 10$, $\pi_2 = 12$.

For nonbasic variables:

(1,2): At upper bound, so $c_{ij} = \pi_j - \pi_i - y_{ij}$

So $y_{12} = \pi_2 - \pi_1 - c_{12} = 12$

(2,3): At upper bound, so $c_{kj} = \pi_j - \pi_k - y_{kj}$

So $y_{23} = \pi_3 - \pi_2 - c_{23} = -13$

(3,4): At upper bound, so $c_{4j} = \pi_j - \pi_3 - y_{34}$

So $y_{34} = \pi_4 - \pi_3 - y_{34} = 12$. 
\( x_{23} \) enters the basis, decreasing from its upper bound.

Choice for leaving variable:
- \( x_{13} \) leaves at upper bound.
- \( x_{24} \) leaves at upper bound.
- \( x_{23} \) leaves at lower bound.

Choose third alternative, so \( x_{23} \) moves from its upper bound to its lower bound, and the basic sequence is unchanged.

New bfs: basic: \( x_{13} = 1 \), \( x_{24} = 1 \), \( x_{41} = 2 \)
Nonbasic: \( x_{12} = 1 \), \( x_{23} = 0 \), \( x_{34} = 0 \).

Find \( \pi_i \) from basic variables:
- Eqs as before since basic sequence unchanged.
- \( \pi_{11} = 0 \), \( \pi_{12} = 12 \), \( \pi_{13} = -2 \), \( \pi_{14} = 10 \).

For nonbasic variables:
- \( (1, 2) \): \( \pi \) unchanged, \( x_{12} \) still at upper bound, so \( y_{12} = 12 \) still
- \( (3, 4) \): \( \pi \) unchanged, \( x_{34} \) still at upper bound, so \( y_{34} = 12 \) still
- \( (2, 3) \): Now at lower bound, so \( y_{23} = 0 \).

Reduced cost: \( \frac{g}{8} c_{23} + \pi_{12} - \pi_{3} = 13 \).

All reduced costs nonnegative, so we are optimum.