The purpose of this assignment is to explore optimization problems using Maple.

Optimization is important in many science and engineering problems. Whenever you are asked to find the "fastest", the "largest", the "smallest", etc. you are finding the maximum or minimum of some quantity. The example below demonstrates how to solve a typical optimization problem using Maple. It is recommended that you practice the example before attempting the exercise.

Example: Find the rectangle of largest area that is contained within the first quadrant, with lower side on the x-axis and left side on the y-axis and that lies within the arc of the ellipse

$$\frac{1}{9} x^2 + \frac{1}{4} y^2 = 1$$

Solution:

First, we need to enter the equation for the ellipse into Maple and solve for $y$ in terms of $x$.

```maple
> eq := (x^2)/9 + (y^2)/4 = 1;

$$eq := \frac{1}{9} x^2 + \frac{1}{4} y^2 = 1$$
```

```maple
> solve(eq, y);

$$\frac{2}{3} \sqrt{-x^2 + 9}, -\frac{2}{3} \sqrt{-x^2 + 9}$$
```

Maple has given two possible answers for $y$. Since we are only looking at a rectangle in the first quadrant, we’ll choose the positive expression.

```maple
> y := %[1];

$$y := \frac{2}{3} \sqrt{-x^2 + 9}$$
```

To see what this portion of the ellipse looks like, let’s graph it.
Our rectangle must be contained within this arc, which means that one of its upper vertices will lie on the curve.

Now that we have established the relationship between \( x \) and \( y \), we need to determine the quantity that is to be maximized, namely the area of a rectangle. We know that the area of a rectangle is \( A = x \cdot y \), so we enter that into Maple.

\[
A := \frac{2}{3} \cdot x \cdot \sqrt{-x^2 + 9}
\]

Now, we are going to maximize the area. We do this by taking the derivative of the quantity and setting the derivative equal to zero.

\[
A' := \frac{2}{3} \cdot \sqrt{-x^2 + 9} - \frac{2}{3} \cdot \frac{x^2}{\sqrt{-x^2 + 9}}
\]

\[
\text{solve}(A' = 0, x);
\]

\[
\frac{3}{2} \sqrt{2}, -\frac{3}{2} \sqrt{2}
\]

This gives us two possible places at which the derivative of the area function is equal to zero. Since we are only looking at the first quadrant, we will choose the positive value.

\[
x_1 := \%[1];
\]

\[
x_1 := \frac{3}{2} \sqrt{2}
\]

\[
x_1 := \text{evalf}(\%);
\]

\[
x_1 := 2.121320343
\]

Apparently, we have a minimum or a maximum at the value \( x = 2.12 \). To verify that it is a maximum, we can try the second derivative test and confirm that it is concave down.

\[
A'' := \text{diff}(A', x);
\]

\[
A'' := \frac{4}{3} \cdot \frac{1}{\sqrt{-x^2 + 9}} - \frac{4}{3} \cdot \frac{x^2}{(-x^2 + 9)^{3/2}}
\]

Now, we test the second derivative at the critical point.

\[
A''(2.12) = \frac{4}{3} \cdot \frac{1}{\sqrt{-2.12^2 + 9}} - \frac{4}{3} \cdot \frac{2.12^2}{(-2.12^2 + 9)^{3/2}}
\]

\[
A''(2.12) < 0
\]

Thus, we have verified that \( x = 2.12 \) is a maximum.
\[ A_{\text{prime}2} := -2 \frac{x}{\sqrt{-x^2 + 9}} - \frac{2}{3} \frac{x^3}{(-x^2 + 9)^{3/2}} \]

\[ \text{subs}(x=x_1, A_{\text{prime}2}); \]

\[-2.666666665\]

The second derivative is negative at that point, so we know it is a maximum.

Next, we need to determine the area of this rectangle by plugging this value of \( x \) into the area equation.

\[ A_{\text{max}} := \text{subs}(x=x_1, A); \]

\[ A_{\text{max}} := 3.000000000 \]

To determine the vertices of the rectangle, we need to determine the corresponding \( y \) value of this \( x \) value.

\[ y_1 := \text{subs}(x=x_1, y); \]

\[ y_1 := 1.414213563 \]

So the vertices of the rectangle are: \((0,0),(0,1.41),(2.12,0),(2.12,1.41)\).

**Assignment:**

Find the area of the largest rectangle that has it's base on the \( x \)-axis and upper vertices on the curve of the parabola \( y=8-x^2 \). (The rectangle lies within quadrants 1&2) List the vertices of the rectangle and sketch the result.