Total Unimodularity

Theory for understanding why the solution to some linear programs are integer. E.g. max flow, assignment problem.

Definition: A square integer matrix \( B \) is called unimodular if its determinant is \( \pm 1 \).

Definition: An m x n integral matrix \( A \) is totally unimodular (TU) if the determinant of each square submatrix of \( A \) is equal to 0, 1, or -1.

Note: If \( A \) is TU then every entry of \( A \) is either 0, 1 or -1.

If \( B \) is formed from a subset of a basis, i.e., each column of \( A \), it determines the basic solution

\[
x = B^{-1} b = \frac{B^{adj} b}{\det(B)}
\]

Thus, if \( \det(B) \neq 0 \) then \( x \) is integer (prov b integer).

S. we have

Theorem: If \( A \) is TU then all the vertices of \( \{x \in \mathbb{R}^n : A x = b\} \)
are integer for any integer vector \( b \).

We also have:

Theorem: If \( A \) is TU then all the vertices of \( \{x \in \mathbb{R}^n : A x \leq b\} \) are integer for any integer vector \( b \). (No proof.)
Theorem The following statements are equivalent:

1. \( A \) is TU
2. \( A^T \) is TU
3. \([A, I]\) is TU
4. A matrix obtained by deleting a unit row (column) of \( A \) is TU
5. A matrix obtained by multiplying a row (column) of \( A \) by \(-1\) is TU
6. A matrix obtained by interchanging two rows (columns) of \( A \) is TU
7. A matrix obtained by multiplying a row (column) of \( A \) by 1 is TU
8. A matrix obtained by a pivot operation on \( A \) is TU.

(Each of these follows straightforwardly from properties of determinants.)
An integer matrix $A$ with $a_{ij} = 0, 1, or -1$ is TU if no more than two nonzero entries appear in any column, and if the rows of $A$ can be partitioned into two sets $I_1$ and $I_2$ such that:

1. If a column has two entries of the same sign, their rows are in different sets.
2. If a column has two entries of different signs, their rows are in the same set.

**Proof**

By induction on size of submatrices.

1x1 matrices: $a_{ij} = 1, -1, 0$ so clear.

$k \times k$ matrices:

Let $C$ be a $k \times k$ submatrix of $A$.

Case (i):

If $C$ has a column of all zeroes, then it is singular, so $\det(C) = 0$.

Case (ii):

$C$ has a column with one nonzero entry:

Expand determinant along that column, result follows from inductive hypothesis.

Case (iii):

Each column of $C$ has two nonzero entries.

Then, $\sum_{i \in I_1} a_{ij} - \sum_{i \in I_2} a_{ij} = 0$ for every column $j$ of $C$.

ie a linear comb of rows is zero. $\therefore \det(C) = 0$. \(\square\)
Corollary. Any LP of the form

\[
\begin{align*}
\min & \quad c^T y \\
\text{subject to} & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]

or

\[
\begin{align*}
\max & \quad c^T y \\
\text{subject to} & \quad A x = b \\
& \quad x \geq 0
\end{align*}
\]

where \( A \) is either

1) the node-arc incidence matrix of a directed graph, or

2) the node-edge incidence matrix of an undirected bipartite graph,

has only integer optimal vertices.

This includes the LP formulations of:

- shortest path,
- max-flow,
- assignment problem,
- weighted bipartite

(No proof.)

Theorem. If \( P(b) = \{ x \in \mathbb{R}^n_+ : A x \leq b \} \) is integral and \( b \in \mathbb{Z}^m \),

which is not empty, then \( A \) is TU.

Note here for expressiveness it is assumed

Interesting theoretical question: determine when \( A \) so that \( \{ x : A x \leq b \} \)

has only integer points.
The following statement are equivalent:

(i) A is TU

(ii) For every $J \subseteq N = \{1, \ldots, n\}$, there exist a partition $J_1, J_2$ of $J$ such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1 \quad \text{for} \ i = 1, \ldots, m.$$

Example: $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

$J = \{1, 2, 4\} \quad J_1 = \{1, 4\} \quad J_2 = \{2\}.

But $J = \{1, 2\}$: No partition.

Interval matrix on TU

Matrix of $0$s and $1$s

In each column, the $1$s appear consecutively, e.g.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Put even rows in $J_1$, odd rows in $J_2.$