Travelling Salesman Problem

Given a graph $G = (V, E)$, a Hamiltonian tour is a cycle that contains all of the nodes. If each edge has a distance $d_e$, the travelling salesman problem is to find the tour with least distance.

In polyhedral terms:

Let $x_e = \begin{cases} 1 & \text{if edge } e \text{ in tour } T. \\ 0 & \text{o/w} \end{cases}$

The TSP is:

\[
\min \sum_{e \in E} d_e x_e
\]

$T$ is a tour.

Christofides heuristic

i) Find a minimum spanning tree $T$ on $G$.

ii) Consider all vertices which have odd degree in $T$.

Find a minimum perfect matching $M$ on these vertices.

iii) Now every vertex has even degree in $M \cup T$ (some edges may be used twice).

Find an Eulerian walk in $M \cup T$ (i.e., a closed walk on which each node appears at least once and each edge appears exactly once).

iv) Short circuit $T$ to get a tour $T$. 
Theorem

Provided the edge weights satisfy the triangle inequality $d_{ij} + d_{jk} \geq d_{ik}$, the Christofides heuristic returns a tour which is no more than 50% from optimality.

Proof

$$d(T) \leq d(W)$$ since short circuit W to get T

$$= d(T) + d(M)$$

Let $\hat{T}$ be an optimal tour.

$\hat{T}$ is connected, so it contains a spanning tree $\hat{T}$, hence $d(\hat{T}) \leq d(T)$.

Consider adding a Hamiltonian path in order, in two different ways.

By doing, $d(\hat{T}) \geq d(M_1) + d(M_2)$

$\therefore d(\hat{T}) \geq \frac{1}{2} \min \{d(M_1), d(M_2)\}$

$\geq d(M)$.

$\therefore d(T) \leq d(\hat{T}) + d(M) \leq \frac{3}{2} d(\hat{T})$.

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2-change heuristic for improving tours

Have two nonconsecutive edges in a tour, and the corresponding four vertices. Consider other ways of using two edges between these vertices. (Careful not to disconnect graph.)

3-change, 5-change:

Generate 2-change to more edges.

Lin showed 3-change very efficient; if start 3-change from a lot of different tours, very likely to get optimal tour.

How do you know when you are optimal?
How far are we from optimal?
Cutting plane algorithm

What constraints describe tours?

- Binary:

  One edge entering and one edge leaving each vertex

  \[ \sum_{e \in \delta(v)} x_e = 2 \]

Are these enough?

No:

\[ \exists \gamma \]

Optimal solution:

\[ \sum_{e \in \delta(v)} x_e \]

\[ \sum_{e \in \delta(v)} x_e = 2 \]

x binary

Need to eliminate subtours.

Every subset \( U \subseteq V \), \( U \neq \emptyset, \phi \), must have at least two edges connecting it to the rest of the graph in any tour.
Get constraints:

\[ \sum_{e \in \mathcal{E}(U)} x_e \geq 2 \quad \forall U \subseteq V, U \neq V, \emptyset. \]

set of edges from \( U \) to \( V \setminus U \).

Now optimal soln to

\[ \min \sum d_e x_e \]

\[ \sum_{e \in \mathcal{E}(U)} x_e = 2 \]

\[ \sum_{e \in \mathcal{E}(U)} x_e \geq 2 \]

\( x \) binary

is optimal tour.

Large number of these constraints add them as cutting planes.

What about LP-relaxation?

\[ \min \sum d_e x_e \]

\[ \sum_{e \in \mathcal{E}(U)} x_e = 2 \]

\[ \sum_{e \in \mathcal{E}(U)} x_e \geq 2 \quad \forall U \subseteq V, \quad \frac{3 \leq |U| \leq |V| - 1}{2} \]

\[ 0 \leq x_e \leq 1. \]
\[ \sum_{\text{ess}} x_e = 2 \text{ in } \{0, 1\} \text{ only} \]

If \( \text{last row or combo of first } n-1 \), would need a odd first row.

But this would give \( 2 \)'s instead of zeros, in all if possible

\[ \sum_{\text{ess}} x_e \geq 2 \text{ facet defining} \text{ for } 3 \leq |U| \leq |V|/2 \]

If we drop one of them we can satisfy all the remaining ones.

But not this one, by taking one subset through \( U \)

and one through \( V \setminus U \).
Solution may not be integral:

\[ d: \]

Opt soln to LP relaxation:

Too small cannot carry out of \([1, 2, 3]\).

C - G case:

Combine adjacency for 1, 2, 3 with weight 1

Combine upper bounds on \(e_4, e_5, e_6\) with weight 4,

\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 4 + \frac{1}{2} \]

\[ = 4. \]

This is an example of a 2-matching inequality:

\[ \sum_{e \in \mathcal{E}(H)} x_e + \sum_{e \in \mathcal{E}} x_e \leq |V| + \left\lceil \frac{|E|}{2} \right\rceil. \]

(eg. Subsets of \(|V| - 1\) edges in \(H\), can only use 2 edges in \(E \setminus \mathcal{E}(H)\)).

Argue why this is valid.
Can generalise to comb inequalities:

\[ |H \cap W_i| \geq 1 \]
\[ |W_i \setminus H| \geq 1 \]
\[ W_i \cap W_j = \emptyset \]

Number of teeth \( W_i \) is odd.

Valid inequality:
\[ \sum_{e \in E(H)} x_e + \sum_{i=1}^{\ell(W_i)} x_{e_i} \leq |H| + \sum_{i=1}^{\ell(W_i)} (|W_i|-1) - \frac{k+1}{2} \]

Fact: exists for a complete graph.

Can be generalised further - see Neuhaus and Zöllig for details.
Well-solvable cases of the TSP (Burke et al., SIAM Review 40 (1998))

Unfortunately (??), even the planar TSP is NP-complete.

1) k-line TSP

Used in, e.g., VLSI design:
Cites, are on k-parallel lines:

Can be solved in \( O(n^k) \) operations,
using a dynamic programming approach.

Result uses the fact that the optimal planar TSP tour does not cross itself, because:

For a rectangle, have:

\[
d(p_1, p_2) + d(p_3, p_4) \leq d(p_1, p_3) + d(p_2, p_4)
\]

2) A cost matrix is a **trapezoidal** matrix if

\[
d_{ij} + d_{rs} \leq d_{ir} + d_{js} \quad \text{if} \quad 1 \leq i < r \leq n
\]

For such a cost matrix, there exists an optimal tour that is **trapezoidal**,

\[
1 \rightarrow i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k \rightarrow j_k \rightarrow \cdots \rightarrow j_1 \rightarrow 1
\]

with \( 1 \leq i_1 < i_2 < \cdots < i_k < n \) and \( n > j_1 > \cdots > j_k > 1 \).

The best trapezoidal path can be found in polynomial time \( (O(n)) \), even though
there are exponentially many such tours.
Vehicle Routing Problem

E.g.: School bus scheduling, mail deliveries, etc.

Have several salesmen visiting the cities.
Each city is visited by exactly one of the salesmen.

So: Need to (1) construct assignment of customers to salesmen
(2) find best route through assigned customers for each salesman. (TSP).

Best algorithms will do (1) and (2) simultaneously.

In practice, for harder than TSP, will usually get 2% away from optimal.

Neuristic: Optimal partitioning: (All vehicles have same capacity and all customers have same demand)

Find out how through

Optimally partition the tour.

Can be twice as bad as true optimal value.

Complications: Customers with varying demands (package sizes)
Vehicles of varying sizes
Time windows for delivery.