The MAXCUT polytope is full-dimensional.

Proof by contradiction.

Assume a constraint \( \sum a_e x_e = b \) satisfied by all the incidence vectors of all cuts. Want to show \( b = 0 \) and \( a_e = 0 \).

1. \( b = 0 \):
   \( x_e = 0 \) \( \forall e \) in the incidence vector of a cut. So \( b = 0 \).

2. \( a_e = 0 \) \( \forall e \).

Pick edge \( \bar{e} = (r,s) \).

(\( r \) and \( s \) are on one side, \( V \setminus \{r,s\} \) on the other):

\[ \sum a_r x_r + \sum_{e \in \bar{e}} a_e = 0 \quad \text{(i)} \]

(\( r \) is on one side, \( V \setminus \{r\} \) on the other):

\[ a_r x_r + \sum_{e \in \bar{e}} a_e = 0 \quad \text{(ii)} \]

(\( r, s \) on one side, \( V \setminus \{r, s\} \) on the other):

\[ \sum_{e \in \bar{e}} a_e + \sum_{e \in \bar{e}} a_e = 0 \quad \text{(iii)} \]

\((i) + (ii) - (iii) \Rightarrow 2a_{rs} = 0 \Rightarrow a_e = 0 \). Thus \( \forall e \in E \).
This proof technique can be extended to prove that an inequality is facet defining.

For simplicity, assume $S$ is full-dimensional.

Let $a^T x \leq b$ be a valid inequality.

Facet definity $\iff$ another valid inequality $g^T x \leq \alpha$

and $(g, \alpha)$ not a multiple of $(a, b)$, and $g^T x = \alpha$ for only $x \in S$ satisfying $a^T x = b$. 

\[ g^T x = \alpha \]

\[ a^T x = b \]

only $x \in S$ satisfying $a^T x = b$
Ej: \( x_{ij} - x_{ik} - x_{kj} \leq 0 \) \((*)\) is a facet defining inequality.

This inequality is satisfied as equality by any \( \{e_{ij}, k\} \) \( \notin \mathcal{E}_k \) with the other vertices assigned randomly.

Assume our inequality is implied by \( g^T x \leq h \).

Consider the empty cut: \( V, \emptyset \). Thus, \( h = 0 \).

Consider the cut \( V \setminus \{l\}, \{l\} \) \((l \neq k)\):
\[
\sum_{v \in V} g_v l = 0 \quad (a)
\]

Consider the cut \( V \setminus \{l, m\}, \{l, m\} \), with \( l, m \notin \{e_{ij}, k\} \):
\[
\sum_{v \in V} g_v l + \sum_{v \in V} g_v m - 2g_{lm} = 0 \quad \Rightarrow \quad g_{lm} = 0 \quad (b)
\]

Consider the cut \( V \setminus \{i, j, l\}, \{i, j, l\} \), with \( l \notin \{e_{ij}, k\} \):
\[
\sum_{v \in V} g_v i + \sum_{v \in V} g_v j - 2g_{ij} = 0 \quad \Rightarrow \quad g_{ij} = 0 \quad (c)
\]

Similarly, \( V \setminus \{i, k\}, \{i, k\} \) with \( k \notin \{e_{ij}, k\} \) where \( j_k = 0 \). \((d)\)

Still need to find \( j_{ik}, j_{ik}, j_{ik} \) and \( j_{ik} \) for \( k \notin \{e_{ij}, k\} \).

Consider \( \sum_{v \in V} i, i = 0 \):
\[
\sum_{v \in V} g_v i = 0 \quad \Rightarrow \quad g_{ij} + g_{ik} = 0 \quad (e)
\]

Similarly, \( \sum_{v \in V} j, j = 0 \):
\[
\sum_{v \in V} g_v j = 0 \quad \Rightarrow \quad g_{ij} + g_{jk} = 0 \quad (f)
\]

Finally, for \( i, k \notin \{e_{ij}, k\} \):
\[
\sum_{v \in V} g_v i + \sum_{v \in V} g_v k - 2g_{ik} = 0 \quad \Rightarrow \quad g_{ik} + g_{jk} = 0 \quad \quad \text{agreement of coefficients.}
\]

Thus, \( g_{ik} = g_{jk} = -g_{ij} \quad (g) \)

\( g_{ik} = g_{jk} = -g_{ij} \quad (h) \)

Finally, \( V \setminus \{i, k\} \Rightarrow 0 = \sum_{v \in V} g_v i + \sum_{v \in V} g_v k - 2g_{ik} - 2g_{ik} - 2g_{ik} = j_{ik} + g_{ik} - 2g_{ik} = -2g_{ik} \quad (k) \)
**MAXCUT problem**

\( G' = (V, E) \), Edge cost \( c_e \).

Divide \( V \) into \( i \) and \( j \) s.t. \( V_i \), \( V_j \) and \( V_i \cup V_j = V \).

\( V_i \cap V_j = \emptyset \).

Want to maximize \( \sum c_e \).

Hold \( x_e = 1 \) if edge is in cut.

\( x_e = 0 \) otherwise

- Every cycle and every cut contains an even number of edges.

Got:

\[ x(F) - x(C - F) \leq |F| - 1 \] (\( \heartsuit \))

for every cycle \( C \), contains subset \( F \) of \( C \), \( |F| \) odd.
How do we find violated constraints?

Heuristic (Beraha, Jünger, Nemhauser, also in for IBM’s LINGO):

Breath First search:

```
  Ven (0)  > side
  seed > other side
```

Grow tree until find vertex we’ve already seen.

Hopefully, this gives a violated inequality.

Exercise (Beraha, Jünger):

Form a new graph $G'$ with node set $V'$, = $V$ for each old vertex $v$;
for each $(i, j) \in E$, add edge $(i', j')$ with weight $x(i, j)$
for each $(i', j')$ in $E'$, add edge $(i, j)$ with weight $x(i, j)$

Find shortest path from $v \rightarrow v'$ to $v''$.
If path has length $< I$, the cycle may be violated.
Example:

Length = 1 + 2 + 2 + 1 + 3 + 0
= 8 
0.7 < 1.
(This may not be the shortest path)

Integer programming formulation:

\[
\text{max } \sum \mathbf{c}_e \mathbf{x}_e
\]

subject to:

\[
\mathbf{x} (F) - \mathbf{x} (C \setminus F) \leq |F| - 1
\]

for every cycle \( C \), subcycle \( F \subseteq C \),

\[
|F| \text{ odd}
\]

\( \mathbf{x} \) binary.

Solution to LP relaxation may not be integer.
Other constraints are needed in the description of \( \text{conv}(S) \).

E.g.: Bicycle wheel:

\[ n+2 \text{ vertices, } n \text{ odd.} \]

Can have at most \( 2n \) of these edges in a cut—center vertices on one side, rim vertices on the other.

If all \( x_{ij} = \frac{1}{3} \) then all odd-set inequalities are satisfied.

We have \( 3n+1 \) edges, so \( x_{ij} = \frac{1}{3} \) has weight \( 2n + \frac{2}{3} > 2n \).

So the constraint \( \sum_{e \in \text{cycle}} x_e \leq 2n \) is valid, and not implied by the odd-set constraints.

Note: If \( G \) contains no \( K_5 \) minor (e.g., \( G \) is planar), then the odd-cycle inequalities suffice.
Linear Ordering

Grötschel, Jünger, Neumüller, 8x 1984

Given $n$ objects, with $c_{ij} = \text{cost of having } i \text{ before } j$,
find the best ordering of the objects.

NP-Complete.

Let $x_{ij} = 1$ if $i$ before $j$.

Can express as:

$$\min \sum_{i,j} c_{ij} x_{ij}$$

s.t.

$$\sum_{i,j} x_{ij} = 1$$

$x$ is decision vector of ordering.

Note that $x_{ij} + x_{ji} = 1$. So decide half.

The variables: Let $x_{ij} = \frac{1}{2} (x_{ij} + x_{ji})$, $1 \leq i < j \leq n$,
Can express as an IP:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{\frac{n}{2}} c_{ij} x_{ij} \quad \text{s.t.} \quad x_{ij} + x_{ji} = 1 \quad \text{for all } i,j \\
\quad x_{ij} + x_{jk} + x_{ki} \leq 2 \quad \text{for all } i,j,k \\
\quad x_{ij} \in \{0,1\}.
\]

The inequalities \( x_{ij} + x_{jk} + x_{ki} \leq 2 \) rule out

\[\begin{array}{c}
 i \\
 \downarrow \\
 j \\
 \downarrow \\
 k
\end{array} \quad \text{impossible for an ordering.}
\]

Can show:

(i) \( \dim (\text{in order polytype}) = \frac{m(m-1)}{2} \) (so only necessary)

(ii) \( x_{ij} \geq 0 \) and \( x_{ij} \leq 1 \) are facets of lin order polytype. This is only a facet if no dud half the vertices.

(iii) \( x_{ij} + x_{jk} + x_{ki} \leq 2 \) is a facet of lin order polytype.

(iv) \( x_{ij} + x_{jk} + x_{ki} + x_{kl} \leq 3 \) is not a facet.

(v) There are other classes of facets.
Proofs of (i), (ii) rest on showing that get right number of affinely independent points.

Do this by only looking at $x_{ij}$ for $1 \leq i < j \leq n$.

Consider the ordering:

\[
\begin{array}{cccc}
6 & 5 & 4 & 3 \\
6 & 5 & 4 & 2 \\
6 & 5 & 4 & 1 \\
6 & 5 & 1 & 4 \\
6 & 1 & 5 & 4 \\
1 & 6 & 5 & 4 \\
1 & 6 & 5 & 3 \\
1 & 6 & 5 & 2 \\
1 & 6 & 2 & 5 \\
1 & 2 & 6 & 5 \\
1 & 2 & 5 & 3 \\
1 & 2 & 6 & 3 \\
1 & 2 & 3 & 5 \\
1 & 2 & 3 & 4 \\
\end{array}
\]

x = 0
\[
\begin{bmatrix}
1 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

Given $5 + 4 + 3 + 2 + 1 + 1 = \frac{5(5+1)}{2}$
og all only ordering is there.

Can't do the last step. Have $x_{65} = 1$. So there are $n(n-1)$ affinely independent points.
With this chain of variables, easy to see all but independence.
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4 exercises for adjusting positions of 5.

In general we get:

\[(m-1) + (m-2) + \ldots + \frac{3+1+2}{2} = \frac{m(m-1)}{2}\]

so here the facet.
\[ x_{12} + x_{23} + x_{31} \leq 2 \]
\[ x_{13} + x_{34} + x_{41} \leq 2 \]
\[ x_{12} + x_{13} + 1 + x_{31} + x_{41} \leq 4 \]
\[ \Rightarrow x_{12} + x_{13} + x_{34} + x_{41} \leq 3 \] \checkmark