Strong with induction for structured edge programs

Motivate by considering 2d packing problem:

Given a graph \( G = (V, E) \), a 2d packing is a subset \( U \subseteq V \)

such that \( \forall u, v \in U \Rightarrow (u, v) \notin E \)

no pair in the set is joined by an edge.

So, setting \( x_i = \begin{cases} 1 & \text{if node } v_i \text{ is in packing} \\ 0 & \text{ otherwise} \end{cases} \)

we have

\[ S = \{ x \in \mathbb{B}^n : x_i + x_j \leq 1 \text{ for all } (i, j) \in E \} \quad (n = |V|) \]

\( S \) contains all 2dly independent vectors:

The origin.

The \( n \) unit vectors.

\( \dim(\text{cone}(S)) = n \)

\( C \subseteq V \) is a clique if each pair of nodes \( v, w \in C \) is joined by an edge.

Any node packing can include at most one vertex for any clique.

So, get valid inequalities:

\( \sum_{i \in C} x_i \leq 1 \text{ for any clique } C \) of \( G \).
A maximal clique is a clique \( C \) such that \( CU \{i\} \) is not a clique for any \( i \in U \setminus C \).

So for example, maximal cliques \( C \) are:

\[
\begin{align*}
    x_1 &+ x_2 + x_3 &\leq 1 \\
    x_1 &+ x_2 + x_4 &\leq 1 \\
    x_1 &+ x_4 + x_5 &\leq 1 \\
    x_1 &+ x_5 + x_6 &\leq 1 \\
    x_1 &+ x_6 &\leq 1
\end{align*}
\]

Maximal clique constraints define facets of \( \text{conv}(S) \):

A facet is a face of dimension \( n-1 \) \( (\dim \text{conv}(S) = n) \).

So we give a slab, indexed by \( j \) in \( S \) which satisfies \( \sum_{j \in C} x_j = 1 \).

For \( j \in C \):

Consider the path going consisting point \( j \).

For \( j \not\in C \):

Since \( C \) is maximal, \( j \cup \{k\} \not\subset C \) such that \( \langle j, k(j) \rangle \in E \).

Consider the path consisting of \( j \) and \( k(j) \).

The incidence vector of \( j \) for the pathings \( \langle j, k(j) \rangle \) is independent, so the clique constraint \( \sum_{j \in C} x_j = 1 \) defines a facet.

For example:

Consider \( C = \{1, 2, 3\} \).

The graph pathings:

\[
\begin{bmatrix}
    1 & 1 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 1 \\
    0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Incidence matrix:

\[
\begin{bmatrix}
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 1 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 1 \\
    0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Now consider point \( x = \frac{1}{5} (0, 1, 1, 1, 1, 1)^T \).

This satisfies all clique constraints, but it is not a cut (5) because it is on the extreme point of polyhedron defined by (5) and non-negative.

To cut off \( x \), consider odd hole inequality.

Suppose \( \exists H \subseteq V \), \( |H| \) odd, vertices of \( H \) can be ordered \((v_1, v_2, \ldots, v_{|H|})\) such that \((v_i, v_{i+1}) \in E\) if \( i \) is odd and \( i \) is even.

Then \( H \) is an odd hole.

If \( H \) is an odd hole, then

\[
\sum_{j \in H} x_j \leq \frac{|H|-1}{2}
\]

is satisfied by all node packings.

So, for example, \( H = \{2, 3, 4, 5, 6\} \) is an odd hole, and we obtain the constraint

\[
x_2 + x_3 + x_4 + x_5 + x_6 \leq 2 \quad (7) (1)
\]

This cuts off \( x \).

Does (7) (1) define a facet? No—it gives a facet of
dimensions.

The packings \( \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 6\}, \{4, 6\} \) satisfy (7) (1) as equality and they give all node vectors. But if we look for a packing that satisfies (7) (1) as equality,
We have a few but not a great.

Maybe (1) can be improved by tilting it to get a facet.

Consider the set

$$2x_1 + (x_2 + x_3 + x_4 + x_5 + x_6) \leq 2. \quad (2)$$

For what values of $x_1$ is the set empty for $S$?

Need to consider what happens when $x_1 = 0$ and $x_1 = 1$.

If $x_1 = 0$, (2) is valid for $S$ for any $x_2$.

If $x_1 = 1$, any packing for $S$, $x_1 = x_2 = x_4 = x_5 = x_6 = 0$.

So (2) is valid for $S$ for $x \leq 2$.

So we get

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 2 \quad (3)$$

valid for $S$.

(3) is facet for $S$: this basic packing, given above, together with (2) define two characteristic vectors which satisfy (3) at equality, and the convex hull of these vectors is empty, independent.

This is an illustration of a general procedure called
Incidence vectors of packing, satisfying
\[2v_1 + v_3 + v_4 + v_6 + \psi = 2.\]

To show they are all real, sufficient to show they are \(i, \bar{i}\) real.

\[\begin{align*}
x_1 & = 0 \\
x_2 + x_3 & = 0 \\
x_2 + x_6 & = 0 \\
x_3 + x_4 & = 0 \\
x_4 + x_5 & = 0 \\
x_5 + x_6 & = 0
\end{align*}\]

\[\sum 1 = \psi + x_4 = -x_2 - x_6 = -1 \Rightarrow l_1 = 0\]
Proposition
Suppose $S \subseteq B$.

Let $S^0 = S \cap \{ x \in B : x_i = 0 \}$

$S^1 = S \cap \{ x \in B : x_i = 1 \}$

Assume $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ (4) is valid for $S^0$.

Let $\gamma = \max \{ \sum_{j=2}^{n} \pi_j x_j : x \in S^1 \}$.

Then
$$\alpha x_i + \sum_{j=1}^{n} \pi_j x_j \leq \pi_0$$

is valid for $S$ for any $\alpha \leq \pi_0 - \gamma$.

Moreover: if $\alpha = \pi_0 - \gamma$ and if $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a face of $S^0$ of dimension $k$,

then $\alpha x_i + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a face of $S$ of dimension $k+1$.

Take $x_i$ as a constant.

Let $x_i$ and $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ define a facet of $S$. Then

$\alpha x_i + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a facet of $S$ provided $\alpha(\pi_0 - \pi_0(\gamma)) > 0$.

NB: Proposition gives another way for finding lifting that does sequentially.

A valid ray defines a facet, show that it can be obtained as a maximum lifting of a facet of a lower dimensional face.
Proof

To show (5) is valid:

If \( x \in S^0 \):
\[
\alpha x_i + \sum_{j=2}^{n} \pi_j x_j = \frac{\pi_i}{\alpha} x_i \leq \pi_0 \quad \text{from (4)}
\]

If \( x \in S^1 \):
\[
\alpha x_i + \sum_{j=2}^{n} \pi_j x_j = x + \sum_{j=2}^{n} \pi_j x_j \leq x + q
\]
\[
\leq \pi_0 - q + q = \pi_0
\]

To show (5) is a face of dimension \( k+1 \) if (4) is a face of \( S^0 \) of dimension \( k+1 \):

Since (4) gives a face of dimension \( k+1 \) of \( S^0 \), \( x \) will satisfy \( \pi_j x_j = \pi_0 \) for some \( j \in \{2, \ldots, n\} \).

Let these points be \( x^1, x^2, \ldots, x^{k+1} \).

Note that \( x^1 = 0 \), since \( x \in S^0 \).

Now consider the point \( x^* \), which solves
\[
\max \{ \pi^T x^* : x \in S^1 \}
\]

Then \( \alpha x_i^* + \sum_{j=2}^{n} \pi_j^* x_j^* = \pi_0 \), and \( x_i^* = 1 \).

Since \( x_i^* = 1 \), \( x^* \) cannot be expressed as a affine combination of \( x^1, \ldots, x^{k+1} \).

Hence the points \( x^*, x^1, \ldots, x^{k+1} \) are affinely independent.
Proof of first part:

(4) define a limit of $S^o$.

Then (3) defines a face of dimension $\geq \dim(S^o) = \dim(S) - 1$ of $S$. Since (4) is a limit of $S^o$, let $x^o \in S^o$ such that $\sum_{j=1}^{\infty} \pi_j x_j < \pi_0$.

Then $x^o \in S$ is such that

$$\alpha x^o + \sum_{j=1}^{\infty} \pi_j x_j < \pi_0,$$

so (5) does not enter $S$, so (5) defines a limit of $S$.

Let any procedure shall be used sequentially:

Have any

$$\sum_{j \in N^1} \pi_j x_j < \pi_0$$

valid for $\forall x \in B^m : x_j = 0, j \in N \setminus N^1$.

Left variable in $N^3 \setminus N^1$ are at a time to get

$$\sum_{j \in N^3 \setminus N^1} \alpha_j x_j + \sum_{j \in N^1} \pi_j x_j < \pi_0$$

valid for $S$.

See over for e.g.
Eq 1: Lift may order matters:

\[ x_1 \cdots x_6 \leq 2 \]

Lift on \( x_1 \) then \( x_2 \) gives \( x_3 + x_4 + \cdots + x_6 \leq 2 \)

Lift on \( x_7 \) then \( x_8 \) gives \( x_9 + \cdots + x_6 + 2x_7 \leq 2 \).

Eq 2: Dimension of faces can increase by more than 1:

\[ x_i \leq 1 \text{ has dimension 0 so} \]

Lift in \( x_i \leq 1 \), until dimension \( 2 \leq 5 \).