04/15/04 More general diffusion processes

Kloeden + Platen Sec. 1.7

Allow the continuous-time, continuous-state stochastic process to have a drift term and diffusion term, both of which can depend on time and on state of the process.

Intuitively: 

Drift: \( v(x, s) = \lim_{t \to s} \frac{1}{t-s} \left| E(X(t) - X(s)) \right|_{X(s)=x} \)

Diffusivity: \( k(x, s) = \lim_{t \to s} E \left( \frac{(X(t) - X(s))^2}{2(t-s)} \right) \mid X(s)=x \)

For Wiener process \( v(x, s) = 0 \), \( k(x, s) = \frac{1}{2} \)
Technically, a diffusion process $X(t)$ taken values in $S = \mathbb{R}$ such that its transition probability density $p(s, x; t, dy)$ satisfies:

a) $\lim_{t \downarrow s} \frac{1}{t-s} \int_{|y-x|>\varepsilon} p(s, x; t, dy) = 0$ for any $\varepsilon > 0$ (excludes discontinuous jumps)

b) $\lim_{t \downarrow s} \frac{1}{t-s} \int_{|y-x|<\varepsilon} (y-x) p(s, x; t, dy) = v(x, s)$

c) $\lim_{t \downarrow s} \frac{1}{2(t-s)} \int_{|y-x|<\varepsilon} (y-x)^2 p(s, x; t, dy) = K(x, s)$

Recall: $P(x, s; t, B) = \text{Prob}(X(t) \in B \mid X(s) = x)$

Diffusion processes can be shown to have a transition probability density $p(s, x; t, dy)$ provided $K(x, s) \geq m > 0$:

$p(x, s; t, B) = \int_B p(x, s; t, y) dy$

$p(x, s; t, dy) = p(x, s; t, y) dy$
Kolmogorov backward equation for a diffusion process.

- One way to solve for prob. trans. density.

**Chapman-Kolmogorov eqn:**

\[ p(s, x; t, y) = \int_{\mathbb{R}} dz \ p(s; y, z) p(u, z; t, y) \]

\[ \int_{\mathbb{R}} \]

To derive backward eqn:

\[ p(s - \Delta s, x; t, y) = \int_{\mathbb{R}} dz \ p(s; y, z) p(s - \Delta s, x; s, z) p(s, z; t, y) \]

Assuming smoothness:

\[ p(s, z; t, y) = p(s, x; t, y) + (z - x) \frac{2}{\partial t} p(s, x; t, y) \]

\[ + \frac{1}{2} \ (z - x)^2 \frac{\partial^2}{\partial x^2} p(s, x; t, y) \]

\[ + \mathcal{O}(z - x)^3 \]

Sub into integral.
\[ p(s-As, x, t, y) = \left( \int_{\mathbb{R}} dz \ p(s-As, x, z) \right) \ p(s, x, t, y) \]
\[ + \left( \int_{\mathbb{R}} dz \ (z-x) \ p(s-As, x, z) \right) \frac{1}{2} \frac{\partial}{\partial x} p(s, x, t, y) \]
\[ + \frac{1}{2} \left( \int_{\mathbb{R}} dz \ (z-x)^2 \ p(s-As, x, z) \right) \frac{\partial^2}{\partial x^2} p(s, x, t, y) \]
\[ + \int_{\mathbb{R}} dz \ O(z-x)^3 \ p(s-As, x, z) \]
\[ \text{with } x \leq \theta \leq z \]

As \( As \to 0 \):
\[ p(s-As, x, t, y) = p(s, x, t, y) + v(x, s) As \frac{\partial}{\partial x} p(s, x, t, y) \]
\[ + k(x, s) As \frac{\partial^2}{\partial x^2} p(s, x, t, y) + o(As) \]
\[ < As \]

\[ \frac{p(s-As, x, t, y) - p(s, x, t, y)}{As} = v(x, s) \frac{\partial}{\partial x} p(s, x, t, y) \]
\[ + k(x, s) \frac{\partial^2}{\partial x^2} p(s, x, t, y) \]
\[ + o(1) \]
Taking $A \rightarrow 0$ limit:

\[- \frac{\partial p(s, x; t, y)}{\partial s} = \nabla (x,s) \frac{\partial p(s, x; t, y)}{\partial x} + K(x,s) \frac{\partial^2 p(s, x; t, y)}{\partial x^2}\]

Kolmogorov backward equation for diffusion process

(Initial data: $p(s, x; t = 0, y) = \delta(x - y)$)

Infinitesimal generator

$A = \nabla (x,s) \frac{\partial}{\partial x} + K(x,s) \frac{\partial^2}{\partial x^2}$

(discretization gives random walk w/ bias)

Multi-dims:

$A = \nabla (x,s) \cdot \nabla + K(x,s) : \nabla \nabla$

Recall that the Kolmogorov backward eqn is also the eqn for which

$u(s, x) = \mathbb{E} \left( f(\zeta(t)) \mid \zeta(s) = x \right)$

satisfies, but with "initial" data

$u(t, x) = f(x)$
Forward Kolmogorov equation for diffusion processes

If by parallel argument by using Chapman-Kolmogorov eqn with times \( s, t, t + \Delta t \), it doesn't work.

Instead consider

\[
p(s, x; t + \Delta t, y) - p(s, x; t, y) =
\]

\[
\Delta t
\]

\[
\int_{\mathbb{R}} dz \ p(s, x; t, z) \left( \frac{p(t, z; t + \Delta t, y) - \delta(z-y)}{\Delta t} \right)
\]

\[
\overset{\text{(-K)}}{ightarrow}
\]

\[
\frac{p(t, z; t + \Delta t, y) - p(t + \Delta t, z; t + \Delta t, y)}{\Delta t}
\]

As \( \Delta t \to 0 \)

\[
\text{backward Kolmogorov }
\]

\[
\left. - \frac{\partial p(s, z; t + \Delta t, y)}{\partial s} \right|_{s = t + \Delta t}
\]

\[
\left( \nabla(z, t + \Delta t) \frac{\partial}{\partial z} + K(z, t + \Delta t) \frac{\partial^2}{\partial z^2} \right)
\]

\[
p(t + \Delta t, z; t + \Delta t, y)
\]
So, as $\alpha \to 0$

$$\frac{\partial}{\partial t} p(s, x; t, y) = \int \nabla \nabla \left[ v(z, t) \frac{\partial}{\partial z} + k'(z, t) \frac{\partial^2}{\partial z^2} \right] \delta(z - y)$$

$p(t, z; t', y)$

$p(t, z; t', y) = \delta(z - y)$

Integrate by parts.

$$\frac{\partial}{\partial t} p(s, x; t, y) = \int \nabla \left[ \frac{\partial}{\partial z} \left( v(z, t) p(s, x; t, z) \right) \right] \frac{\partial^2}{\partial z^2} \left( k'(z, t) p(s, x; t, z) \right) \delta(z - y)$$

$$\frac{\partial}{\partial t} p(s, x; t, y) = -\frac{\partial}{\partial y} \left( v(x, t) p(s, x; t, y) \right)$$

$$+ \frac{\partial^2}{\partial x^2} \left( k'(y, t) p(s, x; t, y) \right)$$

Kolmogorov forward equation

$(Initial data: p(s, x; t = s, y) = \delta(x - y))$
Notice Kolmogorov forward equation has the form \[ \frac{\partial p}{\partial t} = A^* p \]

adjoint

Backward eqn. \[-\frac{\partial p}{\partial s} = A p\]

Recall also that if \( \Phi(y,t) \) is the probability density for the process \[ \text{Prob}(X(t) \in B) = \int_B dy \Phi(y,t) \]

then \( \Phi \) satisfies the Kolmogorov forward eqn.

\[ \langle \Phi(x) | X(t) = x \rangle = \int dy \Phi(y,t) f(y) \]

with \( \Phi(y,s) = s(x-y) \).

The various formulas involving absorption probabilities, integrated payoffs, stationary dists from continuous-time MC all carry over with just changing infinitesimal generator.
But what about path properties of a stochastic process? How simulate?

Think about diffusion processes in terms of stochastic differential eqn.

Simple cases:

I) No diffusion \( K = 0 \)

Kolmogorov backward eqn for \( u(s, \mathbf{x}) = \mathbb{E}[f(\mathbf{X}(t))|\mathbf{X}(s) = \mathbf{x}] \)

\[ \frac{\partial u}{\partial s} = \nabla (\mathbf{x}) \cdot \nabla u \]

\[ u(s=t, \mathbf{x}) = f(\mathbf{x}) \]

Solve: Method of characteristics:

Associated trajectory \( \mathbf{X}(t) \)

Which satisfies: \( \frac{d \mathbf{X}(t)}{dt} = \nabla (\mathbf{X}(t), t) \)

Solve this eqn with \( \mathbf{X}(s) = \mathbf{x} \)

\[ \mathbf{X}(t) = \mathbf{x} + \int_{s}^{t} \nabla (\mathbf{X}(u), u) \, du \]

Then \( u(s, \mathbf{x}) = f(\mathbf{x} + \int_{s}^{t} \nabla (\mathbf{X}(u), u) \, du) \)

Solves the Kolmogorov backward eqn.