Simulating Poisson rv's for Poisson point process

\[ \text{Prob}(Y = n) = e^{-\lambda} \frac{\lambda^n}{n!} \quad \text{for } n \geq 0 \]

Recall that for the Poisson counting process \( X(t) \)

with transition rate \( \lambda \):

\[ \begin{array}{cccc}
0 & T_1 & T_2 & T_3 \\
\rightarrow 0 & \rightarrow & \rightarrow & t
\end{array} \]

\[ \text{Prob}(T_j > t) = e^{-\lambda t} \quad \text{and the } T_j \text{ are i.i.d.} \]

\[ \text{Prob}(X(t) = n) = e^{-\lambda} \frac{\lambda^n}{n!} \]

So to simulate a Poisson rv with mean \( \lambda \):

generate i.i.d. exp dist rv's with pdf

\[ p_T(t) = \lambda e^{-\lambda t} \]

Treat each of these as a \( T_j \) (time to next event.)

Keep simulating until \[ \sum_{j=1}^{n} T_j > 1 \]

Then set \( Y_{\text{sim}} = n-1 \) and this simulated rv will satisfy

\[ \text{Prob}(Y = n) = e^{-\lambda} \frac{\lambda^n}{n!} \]
Continuous-time Markov chains:
Long-time properties

Some can be done using discrete-time MC
techniques by associating to a
continuous-time MC with Inf. gen. $A$
a discrete-time MC with prob. trans,
matrix $\tilde{P} = \frac{A_{ij}}{A}$ for $i \neq j$, $\tilde{P}_{ii} = 0$

except for absorbing states $k$: set $\tilde{P}_{kk} = 0$.

This associated DTMC has each epoch
correspond to $1$ transition of CTMC

\[ X(t) \]

\[ \text{CTMC} \]

\[ X_n \]

\[ n \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots \]

Note: $\tilde{X}_n \neq \tilde{X}_{n+1}$

\[ X_n \]

\[ X_{n+1} \]

\[ X_{n+2} \]

\[ \cdots \]
Transience, recurrence, absorption probs for CTMC $X(t)$ can be decided by looking at associated DTMC $\tilde{X}_n$.

But properties: positive recurrence and expected time to absorption + stationary dist require info about time spent in a state so can't answer from assoc. DTMC.

Stationary distribution $\Pi$ for CTMC:

$$\Pi_i = \text{Prob}(X(t) = i)$$

$$\frac{d}{dt} \Pi = 0 \quad \text{Kolmogorov forward: } \frac{d\Pi}{dt} = \Pi A$$

$$\sum_{i,j} \Pi_j A_{ji} = \Pi_i A_{ij} = \sum_{i,j} \Pi_i A_{ij}$$

This equation has similar interpretation as discrete case, but now flux.

Total flux in = total flux out:

$$\sum_{j\neq i} \Pi_j A_{ji} = \Pi_i A_{ij} = \sum_{i,j} \Pi_i A_{ij}$$

Can often use detailed balance ideas. 

Haken
Positive recurrence for CTMC when recurrent; construct invariant measure \( \mu \) for CTMC by fixing state \( k \) and letting \( X(0) = k \) and

\[
\nu_j = \mathbb{E} \int_0^{T_k(1)} I\{X(s) = j\} \, ds
\]

= expected time spent in state \( j \) between visits to \( k \).

\[
T_k(1) = \min \{ t \geq 0 : X(t) = k \text{ and } X(t-\varepsilon) \neq k \text{ for some } 0 < \varepsilon < t \}
\]

Show: \( \mu(A) = 0 \) (invariant measure)

Note: \( \sum_j \nu_j = \mathbb{E} T_k(1) \)

It is true that: \( \mathbb{E} T_k(1) < \infty \iff \) positive recurrent

\( \iff \) existence of state dist \( \pi = \frac{\mu}{\sum_j \nu_j} \)

\( \iff \pi \) is the limit dist of CTMC.
Expected cost/reward until absorption into a class or state.

Let $\mathcal{D} \rightarrow \mathbb{R}$ (function defined on transient states).

Let $f$ be a function on the state space $S$ of the CTMC $X(t)$.

Want to calculate: $\mathbb{E} \left[ \int_0^T f(X(t)) \, dt \mid X(0) = i \right]$

where $T$ is the first time $X(t)$ enters a certain state $k$ (or class of states),

$f$ is a rate of cost or reward.

Could do this by first step analysis based on first transition time.

But we will instead do it by using Kolmogorov backward equation which makes our argument generalize to the case where $X(t)$ varies continuously (SDE).
To warm up, consider first:

\[ W_i(t) = \mathbb{E} \left[ \int_0^t f(X(s)) \, ds \mid X(0) = i \right] \]

Upper time limit \( t \) is deterministic.

Kolmogorov backward eqn says:

\[ U_i(t) = \mathbb{E} \left[ f(X(t)) \mid X(0) = i \right] \]

satisfies \[ \frac{dU_i}{dt} = A U_i \]

\[ U_i(t=0) = f \]

Formal solution:

\[ U_i(t) = \left( \mathbf{e}^{At} f \right) \]

- Literally a solution if \( A \) is finite-dim matrix,
- if \( A \) is infinite-dim think of this
  as a notation for the solution of
  the eqn,
- semigroup theory

\[ W_i(t) = \int_0^t \mathbb{E} \left[ f(X(s)) \mid X(0) = i \right] \, ds \]

\[ = \int_0^t U_i(s) \, ds \]

\[ W(t) = \int_0^t U(s) \, ds = \int_0^t e^{As} f \, ds \]

\[ = \mathbf{A}^{-1} \left( e^{At} - I \right) f = \mathbf{e} \]

Could do l'Hôpital's rule for matrices

\( A^{-1} \) doesn't really exist (\( A \) may have 0 ev).
More rigorous and simpler:
\[ A \overrightarrow{w}(t) = \int_0^t A e^{A s} \overrightarrow{f} \, ds = \left( \int_0^t \frac{d}{ds} (e^{A s}) \, ds \right) \overrightarrow{f} \]
\[ A \overrightarrow{w}(t) = (e^{A t} - I) \overrightarrow{f} \]
Note that \( \overrightarrow{w}'(t) = e^{A t} \overrightarrow{f} \)

So get autonomous DE for \( \overrightarrow{w}(t) \):
\[ \frac{d\overrightarrow{w}}{dt} = A \overrightarrow{w} + \overrightarrow{f} \quad \text{for } t > 0 \]
\( \overrightarrow{w}(t=0) = \overrightarrow{0} \)

where \( W_i(t) = |E \left[ \int_0^t f(X(s)) \, ds \mid X(0) = i \right] \]

This formula generalizes to case of SDE's.

Now consider
\[ V_i = |E \left[ \int_0^{\tau_k(1)} f(X(t)) \, dt \mid X(0) = i \right] \]

where \( \tau_k(1) \) is first hitting time of state \( k \),

Assume \( f \) is bounded and that \( \text{Prob}(\tau_k(1) \leq t) = 1 \)

We will set \( f(x)=1 \) and make \( k \) an absorbing state.
\( f(x) = f(i) \) for \( i \neq k \)
\( f(k) = 0 \)
The modified in f. gen w/ k a trapping state is defined:

\[ \hat{A}_{ij} = A_{ij} \quad \text{for } i \neq k \]

\[ \hat{A}_{kj} = 0 \]

\( X(t) \) is the CTMC w/ inf gen \( \hat{A} \)

Then:

\[ \int_0^\infty \mathbb{I}_k(1) f(X(t)) \, dt = \int_0^\infty \mathbb{I}_k(1) \hat{f}(X(t)) \, dt = \int_0^\infty \hat{f}(X(t)) \, dt \]

Apply prev result to last expression:

\[ \hat{A} \, \mathbb{1} = \lim_{t \to \infty} (e^{\hat{A}t} - I) \xrightarrow{p} \]

\[ \lim_{t \to \infty} e^{\hat{A}t} \xrightarrow{p} = \lim_{t \to \infty} \hat{P}(t) \xrightarrow{p} = 0 \text{ w/ prob } 1 \]

because

\[ \left| \sum_{j \neq k} \hat{P}_{ij}(t) \frac{f_j}{\hat{f}_j} \right| = \left| \sum_{j \neq k} \hat{P}_{ij}(t) \right| \leq \| f \|_{\infty} \sum_{j \neq k} \hat{P}_{ij}(t) \to 0 \text{ w/ prob } 1, \text{ since } \mathbb{I}_k(1) \rightarrow 0 \text{ w/ prob } 1. \]
So \( \hat{A} \hat{V} = - \hat{v} \)

Rewrite this as:

\[
\begin{align*}
\hat{v}_j &= - f(j) \quad \text{for } j \neq k \\
\text{with } \hat{v}(k) &= 0
\end{align*}
\]

This generalizes to SDEs.

Here \( V_i = \mathbb{E} \left[ \int_0^{T_i} f(X(t)) \, dt \mid X(0) = i \right] \)

This generalizes directly for calculating

\( \mathbb{E} \left[ \int_0^T f(X(t)) \, dt \mid X(0) = i \right] \)

for \( T = \text{first hitting time of class of states} \)

or even for any Marker time \( T \).

- modify MC by introducing another state that is entered when the Marker time is reached.
- 
  \( \text{converges Marker time } T \) to
  \( \text{first hitting time of a state} \) in extended MC

Optimal stopping theory: [Lauwer Ch. 4]: dynamic programming

- tries to find optimal Marker time \( T \)
Examples for long time properties of CTMC:

1) Birth-death processes: State space $S = \mathbb{R}_\geq 0$.

$$A_{i, i+1} = \lambda_i \quad \text{for } i \geq 0$$

$$A_{i, i-1} = \mu_i \quad \text{for } i \geq 1$$

$$A_{i,j} = -\overline{A}_i = -(\lambda_i + \mu_i)$$

$$A_{i,j} = 0 \quad \text{for } |i-j| > 1$$

Invariant measure: $\overrightarrow{\pi}$

$\overrightarrow{\pi} A = 0$.

Same kind of calculation as for DTMC birth-death chain.

Detailed balance works:

$$\nu_j = \frac{\overrightarrow{\pi}_{i+1}}{\overrightarrow{\pi}_i} \lambda_i$$

Start dist exists if $\sum_{j=0}^{\infty} \nu_j < \infty$

$\overrightarrow{\pi} = \frac{\overrightarrow{\pi}}{\sum_{j=0}^{\infty} \nu_j}$

Can study transience in some way (Lafler, See. 3.3)
Consider a special case of birth death process consisting of single-server queue with
\[ \lambda_i = \lambda \text{ for } i \geq 0 \text{ (rate of arrival of demand)} \]
\[ M_i = M \text{ for } i \geq 0 \text{ (rate of completing service)} \]

\[ n_j = \left( \frac{\lambda}{M} \right)^j \sum_{j=0}^{\infty} n_j = \frac{1}{1-\frac{\lambda}{M}} \text{ if } \lambda < M \]

\[ \Pi_j = \left( 1 - \frac{\lambda}{M} \right) \left( \frac{\lambda}{M} \right)^j \text{ geo dist. for } j < M \]

If \( \lambda > M \): transient
\( \lambda = M \): null recurrent

Focus on positive recurrent case \( \lambda < M \).

What is the avg # requests waiting:

\[ \langle X(t) \rangle = \sum_{j=0}^{\infty} j \Pi_j \text{ for large } t \]

\[ = \sum_{j=0}^{\infty} j \left( 1 - \frac{\lambda}{M} \right) \left( \frac{\lambda}{M} \right)^j \]

\[ = \frac{\lambda}{M - \lambda} \]

Using \( \sum_{j=0}^{\infty} j x^j = x \frac{d}{dx} \sum_{j=0}^{\infty} x^j = x \frac{d}{dx} \frac{1}{1-x} = \frac{x}{(1-x)^2} \)