03/04/04  Cost/reward sums in transient states of Finite State, Discrete Time Markov Chains

Functions of MC. $f : S \rightarrow \mathbb{R}$

$\frac{1}{n} \sum_{j=1}^{n} f(X_j)$ behave like?

Long run: use limit distribution of the recurrent classes. You can get absorbed into

Also of interest:

$\sum_{n=0}^{T-1} f(X_n)$ where $T$ is the first time that the set of transient states is left.

$T = \min \{ n \geq 0 : X_n \notin T \}$
Particular interest.

\( a) \ f(i) = 1 \Rightarrow \text{the sum of steps until hit a recurrent class} \)

\( b) \ f_{ij} = \delta_{ij} \Rightarrow \text{the sum of epochs spent in state } i \text{ before hit a recurrent class} \)

We'll focus on expected values:

\[
W_i = \mathbb{E} \left( \sum_{n=0}^{T-1} f(X_n) \mid X_0 = i \right)
\]

First step analysis:

\[
W_i = f(i) + \mathbb{E} \left( \sum_{n=1}^{T-1} f(X_n) \mid X_0 = i \right)
\]

\[
= f(i) + \sum_{j \in S} \mathbb{E} \left( \sum_{n=1}^{T-1} f(X_n) \mid X_0 = i, X_1 = j \right) \cdot \text{Prob}(X_1 = j \mid X_0 = i)
\]

(used: \( \mathbb{E}(Y \mid A) = \sum_{j \in S} \mathbb{E}(Y \mid A, Z = j) \cdot \text{Prob}(Z = j \mid A) \))

Related to \( \mathbb{E} f(X) = \sum_{j \in S} f(j) \cdot \text{Prob}(X = j) \)
\[ W_i = f(i) + \sum_{j \in S} \text{IE} \left( \sum_{n=1}^{\infty} f(X_n) \mid X_1 = j \right) \text{Prob}(X_1 = j \mid X_0 = i) \]

By Markov property

\[ W_i = f(i) + \sum_{j \in S} W_j p_{i,j} \]

In terms of matrices

\[ \vec{W} = \vec{f} + \vec{Q} \vec{W} \]

where \( \vec{Q} \) is submatrix of \( \vec{P} \) corresponding to transient states.
\[ \mathbf{w} = \sum_{n=0}^{\infty} \mathbf{Q}^n \mathbf{f} \]

Recall that \((\mathbf{Q}^n)_{ij} = \Pr(b(X_n = j | X_0 = i)\)

\((\mathbf{Q}^n)_{ii}\) = expected return at n th step, if start from i.

Special cases:

\[ (I - \mathbf{Q}^{-1} \mathbf{1})_{ii} = \mathbb{E} (T | X_0 = i) \]

\[ (I - \mathbf{Q})^{-1}_{ij} = \text{expected amount of time spent in state } j, \text{ given that } X_0 = i \]

Remarks: 1) Calculate Variances of \( T \)
\[ \sum_{n=0}^{\infty} f(X_n) \] using similar technique

Bhat + Miller, Elements of Applied Stochastic Processes
Sec. 2.1 + 2.6.
Derviation flawed, result OK.
Apply it in credit management.
Remark 2): Use similar techniques to answer questions like:

a) Starting from state $i$, what is the expected amount of time until I hit state $j$?

b) Starting from state $i$, how many times on avg do I visit state $k$ before visiting $j$?

Can do this even if $i, j$ are arbitrary elements of $\mathcal{M}_C$.

Consider a modified $\mathcal{M}_C$ where $j$ is made an absorbing state.

Then apply formulas to modified $\mathcal{M}_C$ where $i$ is now transient if $i \to j$ in original $\mathcal{M}_C$. 
This concludes finite state MC
All basic questions stated.
Now enter stochastic models
Part 2
\[ \begin{align*}
\text{infinite states (but discrete, countable)} \\
\text{continuous time}
\end{align*} \]
Part 3
\[ \begin{align*}
\text{continuous state space}
\end{align*} \]

Countable State, Discrete Time MC's
- infinitely many states; \( S = \mathbb{R}_{\geq 0} \) or \( S = \mathbb{R} \), but discrete
- focus on time-homogenous case

References: Lawler Ch. 2
Koralj + Taylor Ch. 2 + 3

Examples:
1) Lifetime of products
2) Distribution of \( \Theta \)-clumped flaws in a product
3) Capital or asset models
4) Populations
5) Queues
6) Random walks on unbounded graphs
Let's re-examine concepts from finite state MC — what carries over and what needs modification?

Finite-time formulas (Chipman-Kolmogorov etc.)
- Work the same way
- More probability transition matrix \( P \) is infinite-dimensional

\[
\vec{\Omega}^{(n)} = \text{Prob}(X_n = j)
\]

\[
\vec{\Omega} = \vec{\Omega} P^n
\]

\[
\vec{\Omega}^{(n)} = \sum_{k \in S} \vec{\Omega}^{(w)}(P^n)_{kj}
\]

\[
(P^2)_{ij} = \sum_{k \in S} P_{ik} P_{kj}, \text{ etc.}
\]

Only worry... do infinite sums over \( S \) converge?

Yes because row sums are \( 1 \), and probabilities all bounded by \( 1 \).
Long-time properties? Transience + recurrence.

One new possibility for \( \infty \)-state MC
is to have an irreducible MC
which is transient.

First return to
Recurrence: 2 types of state \( j \).

Positive recurrence: \( \mathbb{E}[T_{j}(1) | X_0 = j] < \infty \)
- expected time to return < \( \infty \)

Null recurrence: \( \mathbb{E}[T_{j}(1) | X_0 = j] = \infty \)
- slowly decaying PDF for \( T_{j}(1) \)

but \( \text{Prob}(T_{j}(1) < \infty) = 1 \).

Recall transience of state \( j \) \( \iff \)
\( \text{Prob}(T_{j}(1) < \infty) < 1 \).

1) What happens in MC of these 3 types?
2) How do I tell what kind of
MC I have?
The two mathematical approaches to these are:

First approach is useful if you can get explicit formulas for \( p^n \) for arbitrary \( n \)
- random walks (Karlin + Taylor Sec. 2.6)
- Karlin + Taylor Sec. 2.5
  Resnick Sec. 2.6

Develop a recursion formula linking
\[
\text{Prob}(X_n = j \mid X_0 = i) \to \text{Prob}(X_n = j \text{ for the first time} \mid X_0 = i) \uparrow \text{at time } n
\]
First passage probability

Develop probability gen fns... get some results:

A) State \( i \) is transient \( \iff \sum_{n=0}^{\infty} (p^n)_{ii} < \infty \)

B) State \( i \) is recurrent \( \iff \sum_{n=0}^{\infty} (p^n)_{ii} = \infty \)

C) If \( T_j = \# \text{ epochs spent in state } j \)
   Then if \( j \in C_k \) (a recurrent class)
   \[ \text{and then } \text{Prob}(T_j = \infty \mid X_0 = i) = 1 \]
   if \( i \in C_k \)
If \( j \notin T \) (transient)
then \( \mathbb{E} (T_j | X_0 = i) < \infty \) for any \( i \in S \)

If \( X_0 = j \), then \( T_j \) obeys a geometric dist.
If \( X_0 \neq j \), then \( T_j \) obeys a similar dist.

Other approach does not require us to know \( p^n \).

First we'll look at what happens in each type of MC, then we'll figure out how to tell what kind of MC a given model is.
1. A) Suppose we have an irreducible MC which has \( \geq 1 \) positive recurrent state \( j \).

Then the whole MC is positive recurrent and has a unique stationary distribution \( \pi \):

\[
\pi_j > 0, \quad \sum_{j \in S} \pi_j = 1, \quad \pi \cdot P = \pi
\]

Also, \( \pi \) acts as a limit distribution

\[
\lim_{n \to \infty} (P^n)_{ij} = \pi_j
\]

\[
\lim_{n \to \infty} \varnothing \cdot P^n = \pi
\]

Proof? Our proof for finite state MC works, built on the hypothesis

\[
\pi_j = \frac{1}{E(\tau_j(1) | X_0 = j)}
\]

which is proven to be true. That proved existence, uniqueness of \( \pi \).

Limit? Can't use Perron-Frobenius for \( \infty \)-dim matrix

Coupling argument works.
B) Recurrent (but not nec. posit. recurrent)
- then what survives from finite time proof is existence of invariant measure

\[ \nu : \nu_j \geq 0 \]
\[ \nu \cdot \vec{p} = \vec{r} \]

Invariant measure is unique up to constant multiple:
If \( \vec{m}, \vec{r} \) are invariant measures, then
\[ \vec{m} = c \vec{r} \]
for some \( c \geq 0 \).

Positive recurrent:
\[ \lim_{n \to \infty} (p^n)_{ij} = \prod_j \lim_{n \to \infty} \vec{\pi} \cdot p^n = \vec{\pi} \]

Null recurrent:
\[ \lim_{n \to \infty} (p^n)_{ij} = 0 \]