Math 2400, Differential Equations: Exam #1
Tuesday February 13, 2001

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The exam will be graded out of 100, in order to receive full credit, you must neatly show all of your work. There are 7 questions on this exam, its best to work though them sequentially. You are not allowed to use calculators or notes of any sort. Just pencils, pens and your own brain. You have 90 minutes.

1. Show that the following equation is exact and then solve it.

\[(y^2 \cos(x) - 3x^2 y - 2x) \, dx + (2y \sin(x) - x^3 + \frac{1}{y}) \, dy = 0\]

with initial condition

\[y(0) = e\]

\[f(x,y) = 0\]

\[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\]

\[= e\]

so exact is true

2. Find the general solution to the equation

(10 points)

\[\frac{dy}{dx} - y = \sinh(x) \quad y(0) = 1\]

\[\frac{d(e^{-x} y)}{dx} = \frac{1}{2} - e^{-2x}\]

\[(+2)e^{-x} y = e^{-x} + \frac{y}{2} + c_1\]

\[(+1) y(x) = \frac{e^{x}}{2} + \frac{e^{-x}}{4} + c_1 e^{x}\]

\[e^{x = 0} \quad y = 1 \quad c_1 = \frac{3}{2} (+1)\]

\[(+1) y(x) = \frac{e^{x}}{2} + \frac{e^{-x}}{4} + \frac{3}{2} e^{x}\]
\[ \frac{\partial f}{\partial x} = y^2 \cos x - 3x^2 y - 2x \quad (+1) \]

\[ f = y^2 \sin x - x^3 y - x^2 + g(y) \quad (+1) \]

Take deriv w.r.t. \( y \)

\[ \frac{\partial f}{\partial y} = 2y^2 \sin x - x^3 - \frac{\partial g}{\partial y} \quad (+1) \]

Compare w/ equation

\[ \frac{\partial g}{\partial y} = \frac{1}{y} (11) \Rightarrow g = \ln y + c \quad (+1) \]

\[ f(x, y) = 0 = y^2 \sin x - x^3 y - x^2 + \ln y + c \quad (+1) \]

When \( x = 0 \) \( y = e \)

\[ 0 = 1 + c \Rightarrow c = -1 \quad (+1) \]
3. Verify that \( y_1 \) is a solution to the differential equation (5pts), find the second solution to the equation (10 pts) and show that their Wronskian does not vanish (5pts).

\[
4x^2y'' + y = 0 \quad y_1 = x^\frac{1}{2} \ln(x) \quad \text{(long)}
\]

(20 points total)

\[
y_1' = \frac{1}{2} x^{-\frac{1}{2}} \ln x + x^\frac{1}{2} \quad y_1'' = \frac{1}{4} x^{-\frac{3}{2}} \ln x + x^{-\frac{1}{2}}
\]

\[
y_1'' = -\frac{1}{4} x^{-\frac{3}{2}} \ln x + x^{-\frac{1}{2}} - \frac{1}{2} x^{-\frac{3}{2}} \ln x - \frac{1}{2} x^{-\frac{3}{2}}
\]

\[
y_1'' = -\frac{1}{4} x^{-\frac{5}{2}} \ln x
\]

Second solution \( y_2 = u \cdot y_1 \)

\[
y_2' = u' y_1 + u y_1'
\]

\[
y_2'' = u'' y_1 + 2 u' y_1' + u y_1''
\]

\[
y_2'' = u'' y_1 + 2 u' y_1' + u y_1''
\]

\[
y_2'' = u'' y_1 + 2 u' y_1' + u y_1'' \quad y_1'' = 0
\]

Set \( u = \ln x \)

\[
u' y_1 + 2 u y_1' = 0
\]

\[
u' = -\frac{2u y_1'}{y_1}
\]

\[
\int \frac{du}{u} = -\int \frac{2x^{-\frac{1}{2}} \ln x}{x^\frac{1}{2} \ln x} \, dx
\]

\[
= -\int \frac{dx}{x} - 2 \int \frac{dx}{x \ln x}
\]

Set \( y_2 = \ln x \)

\[
\frac{dy_2}{y_2} = \frac{dx}{x}
\]

\[
\ln u = -\ln x + c - 2 \int \frac{dy_2}{y_2}
\]

\[
= \ln x + c - 2 \ln x
\]

\[
= \ln x + c - 2 \ln x
\]

\[
\overline{\text{over}}
\]
\[ u = x^{-1} \frac{d}{dx}( x^n ) \]

\[ u = \frac{1}{x} \left( \frac{1}{(2n)x^2} \right) \]

\[ \frac{dV}{dx} = \frac{1}{x} \frac{1}{(2n)x^2} \]

\[ V = \int \frac{dx}{x(2n)x^2} + C_2 \]

Set \[ z = e^{-x} \]

\[ dz = \frac{dx}{x} \]

\[ V = C_1 \int \frac{dz}{z^2} + C_2 \]

\[ = -C_1 \frac{1}{z} + C_2 \]

\[ = -C_1 e^x + C_2 \]

So \[ y = -y_1 - C_1 x^{\frac{1}{2}} + C_2 x^{\frac{1}{2}} e^x \]

But we can disregard the \( C_1 \) term as it is proportional to \( y_1 \)

\[ W(y_1, y_2) = y_1 y_2 - y_2 y_1 \]

\[ = x^2 \left( \frac{d^2}{dx^2} + 1 \right) x^{\frac{n}{2}} - \frac{1}{2} x^{-\frac{n}{2}} x^{\frac{n}{2}} e^x \]

\[ \equiv 1 \]
4. This question is similar to a problem we discussed, but DIFFERENT. It is the differential equation version of what was a difference equation problem. Person A invests $2000 per year on a continuous basis (i.e. a bit every day) into an account which earns 10% interest. This continues for 10 years after which person A stops adding money and allows the account to accrue interest for the next 30 years (retirement).

Person B simply invests $2000 continuously (i.e. a bit every day) for 30 years and it earns 10% interest.

(a) Write a differential equation describing the evolution of A's account value for the first 10 years and solve it (with initial amount = 0). What is its value after 10 years? (10 pts) You can use the fact that $e \approx 2.8$ and $e^3 \approx 21.6$.

(b) Write the evolution equation of the account value for the next 30 years, solve it and determine the final value of A’s account. (5 pts)

(c) Write and solve the evolution equation for B’s account value. (5 pts)

(d) Determine which person has more money at the end and compare their total return on their total investment. (5 points bonus) (Hint, take the ratio of final value to total deposited)

(20 points total + 5 bonus points)

(a) Let $A(t)$ be the amount of A’s account as continuous interest at a rate $r$ is.

$$\frac{dA}{dt} = rA$$

Continuous investing amounts $P$ per year is

$$\frac{dA}{dt} = P$$

$$A(t) = \left[ e^{rt} \frac{P}{r} \right]$$

$@ t=0 \quad A=0\tag{4}$

$A(t) = \frac{P}{r} (e^{rt} - 1)$

$r = 10\% = 0.1$

$P = 2000$

After 10 years

$A(10) = \frac{20000}{0.1} \left( e^{10} - 1 \right) (10)$

$= 20000 \left( 1.8 \right) (10)$

$= 360000 \left( 10 \right)$

$A(10) = \$360000 \left( 10 \right)$
(b) \[ \frac{dA}{dt} = rA \Rightarrow (4) A(t) = A_0 e^{rt(t+1)} \]

\[ A(t) = 36000 \cdot e^{(21,6)} \]

\[ \approx 36000 \cdot 2.2 \]

\[ = 792000 \]

\[ \approx 792000 \text{ (t+1)} \]

(c) \[ \frac{dB}{dt} = rB + P(t+1) \]

\[ B(t) = \frac{P}{r} (e^{rt} - 1)(t+1) \]

\[ t = 30 \text{ (t)} \]
\[ r = 0.1 \]
\[ e^{rt} = e^{3} \]

\[ = 20000 \cdot (e^{3} - 1)(t+1) \]

\[ = 20000 \cdot (20.6) \]

\[ \approx 20000 \cdot 21 \]

\[ = 420000 \]

\[ \approx 420000 \text{ (t+1)} \]

(d) A has more money than B(t+1)

B invests \[ 30 \cdot 20000 = 600000 \text{ (t1)} \]

A invests \[ 10 \cdot 20000 = 200000 \text{ (t1)} \]

Rate of return for A: \[ \frac{292000}{20000} = \frac{292}{20} \approx \frac{80}{20} \]

Rate of return for B: \[ \frac{420000}{60000} = \frac{420}{60} = 7 \text{ (t1)} \]
5. A classic model for the growth of a population is:

\[
\frac{dp}{dt} = rp - ap^3
\]

Sketch the slope vector field plots, and overlay some solutions for the four cases of (i) \( a > 0, r > 0 \), (ii) \( a < 0, r > 0 \), (iii) \( a > 0, r < 0 \), (iv) \( a < 0, r < 0 \) Interpret each with regard to equilibria and their stability.

(15 points)

(i) \( a > 0, r > 0 \)

\[
\frac{dp}{dt} = 0 = rp - ap^3
\]

\[
r = ap^2
\]

\[
p = \frac{\sqrt{\frac{r}{a}}}{2}
\]

\( \therefore \) \( p = 0 \) 

\( \therefore \) equilibria @ \( p = 0 \) unstable (+1) 

\( p = \pm \sqrt{\frac{r}{a}} \) stable (+1)

(ii) \( a < 0, r > 0 \)

\[
\frac{dp}{dt} = 0 = rp + |a|p^3
\]

\( p = 0 \)

\[\text{on} \quad 0 = r + |a|p^2 \quad (\text{no solution})\]

\( \therefore \) only one equilibrium at 

\( p = 0 \) is unstable (+1)
(iii) $a > 0$ $r < 0$

$$\frac{dp}{dt} = -|r|p - a p^3$$

No solution

$$\implies p = 0 \quad (\text{ti})$$

So $p = 0$ equilibrium is stable, no others

(iv) $a < 0$ $r < 0$

$$\frac{dp}{dt} = 0 = -|r|p + |a| p^3$$

$p = 0$

$$|r| = |a| p^2$$

$$p = \pm \sqrt{\frac{|r|}{|a|}} \quad (\text{ti})$$

So $p = 0$ is stable equil ($\text{ti}$)
6. The force that a springboard exerts on itself is given by

\[ F = -m(13x + 6v) \]

where \( m \) is the mass of the board tip, \( x \) is the displacement of the end of the board in centimeters and \( v \) its velocity in centimeters per second. If a diver's jump sets the board tip moving at \(-20\text{ cm/s}\) with an initial displacement of \(-20\text{ cm}\) find the motion of the board tip with time. (15 points)

Now \[ a = \frac{F}{m} \tag{12} \]

\[ \frac{d^2x}{dt^2} = -13x - 6v \tag{+2} \]

\[ \frac{dx}{dt} = -13x - 6v \tag{+1} \]

\[ \frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 13x = 0 \tag{+2} \]

\[ a = e^{3t} \tag{+3} \]

\[ 2^2 + 6 \cdot 2 + 13 = 0 \tag{+2} \]

\[ 7 = -6 \pm \sqrt{36 - 52} = -3 \pm \sqrt{-16} \]

\[ 2^2 = -3 + 2i \quad 2 = -3 - 2i \tag{+2} \]

\[ a(t) = A e^{2it} + B e^{-2it} \]

\[ a(t) = e^{-3t} \left[ A e^{2it} + B e^{-2it} \right] \]

but I know by now that I can rearrange the stuff in parentheses to give some other form of \( a(t) = \cos (2t) \)

\[ a(t) = e^{-3t} \left[ C \sin (2t) + D \cos (2t) \right] \tag{+2} \]
Now we need

\[ x(0) = -20 \]
\[ x'(0) = -20 \]
\[ x(0) = D = -20 \]

\[ x'(0) = e^{-3t} \left( -37 \left[ C \sin(2t) + D \cos(2t) \right] \right) \]
\[ - e^{-3t} \left[ 2 C' \cos(2t) - 2 D \sin(2t) \right] \]

\[ x'(0) = -3D + 2C' = -20 \]

\[ 2C = 3(-20) - 20 \]
\[ 2C = -80 \]
\[ C' = -400 \]

\[ x(t) = -e^{-3t} \left[ 40 \sin(2t) + 20 \cos(2t) \right] \]
7. Solve the following differential equations. (5 pts each)

(a)
\[ tx \frac{dx}{dt} = 3x^2 + t^2 \quad \text{with} \quad x(-1) = 2 \]

(b)
\[ \frac{dy}{dx} = -\frac{3y^2 + 2xy}{4y^2 + 6xy} \]

(Hint for, substitute another function for the dependent variable, chosen carefully.)

(a)
\[
\begin{align*}
t x \frac{dx}{dt} &= 3x^2 + t^2 (t+1) \\
\frac{t}{2} \frac{d(x^2)}{dt} &= 3x^2 + t^2 (t+1) \\
\text{set} \quad u &= x^2 (t+1) \\
(+) \frac{du}{dt} &= 6u + 2t \quad \left< \text{first order linear} \right> \\
(+) \quad u(t) = e^{\int \frac{6}{t} dt} = t^{6} \\
(t+u)' &= 2t^{-5} \quad \Rightarrow \\
(u(t))' &= \frac{1}{2} t^{2} + c t^{-5} \quad \Rightarrow \\
\text{sol + int. const.} \quad y = -\frac{1}{2} c + c (t^{5})^{1/2} \\
\text{when} \quad t=-1 \quad x=2 \quad a=4 \\
(u(t)) = -\frac{1}{2} t^{2} + \frac{4}{t^{5}} \quad \Rightarrow \\
x(t) = \frac{1}{2} t^{2} + \frac{4}{t^{5}} \quad \Rightarrow \\
4 = -\frac{1}{2} + c \quad c = \frac{9}{2} \\
\end{align*}
\]

(b)
\[
\begin{align*}
\frac{dy}{dx} &= -3y + 2x (t+1) \\
x \frac{du}{dx} + u &= -\frac{3u+2}{4u+6} (t+1) \\
(+) \frac{du}{dx} &= -\frac{3u+2}{4u+6} - u \quad \Rightarrow \\
x \frac{du}{dx} &= -\frac{3u+2 - u(4u+6)}{4u+6} \\
x \frac{du}{dx} &= -\frac{4u^2 - 9u + 2}{4u+6} \quad (t+1)
\end{align*}
\]