MATH 2400
Introduction to Differential Equations

Basic Definitions and Terminology

1. Differential Equation

An equation containing derivatives is called a differential equation.

Notation: D.E.

a) An equation involving only ordinary derivatives is called an ordinary differential equation. Notation: O.D.E.

b) An equation involving partial derivatives is called a partial differential equation. Notation: P.D.E.

c) We shall consider an equation having the form:

\[ \frac{dX}{dt} = AX + G, \]

where \( X \) is a vector, \( A \) is a given matrix and \( G \) is a vector valued function. This type of equation is called a system of differential equations.

2. Order

The highest order of derivation is called order of the D.E.

Examples

a) \( y' + y = 0 \) is a first order O.D.E.

b) \( x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \) is a first order P.D.E.

c) \( \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \) is a second order P.D.E.

d) \( y'' - y = 0 \) is a third order O.D.E.
e) \[ \begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases} \text{ is a system of O.D.E} \quad (\text{of order one})

3.1 \textbf{Solution}

Suppose that \( \phi \) is continuously differentiable up to order \( n \), on an interval \( I \), and satisfies an O.D.E. (E) of order \( n \), then \( \phi \) is called a \textbf{solution} of E.

**Examples**

a) \( y' + y = 0 \quad (E) \)
   Clearly for each constant \( c \),
   \[ \phi(x) = ce^x, \quad -\infty < x < \infty \]
   is a solution of (E)

b) \( y'' + \omega^2 y = 0, \quad \omega > 0 \)
   Since \( (\cos \omega x)' = -\omega \sin \omega x, \quad \cos \omega x \) is a solution.
   Similarly \( \sin \omega x \) is also a solution.

**General solution**

A formula describing all solutions of a D.E. is called the general solution of the equation.

**Particular solution**

A solution of a D.E., which is free from arbitrary constants is called a particular solution.
Example: \( y' + y = 1 \)

Clearly, the constant 1 satisfies the D.E. It is a particular solution. But all solutions have the form: \( y(x) = ce^{-x} + 1 \), where \( c \) is an arbitrary constant. Therefore \( y_0(x) = 1 \) is a particular solution and \( ce^{-x} + 1 \) is the general solution.

**Linearity**

An \( n \)th order O.D.E. having the form:
\[
 y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x) y = g(x) \quad , \quad x \in I
\]

where \( a_1, \ldots, a_n \) and \( g \) are functions of \( x \) only, is called linear. Otherwise, the equation is said to be nonlinear.

Examples:

a) \( y'' + x^{-1}y' + y = 0 \), \( x > 0 \), is linear
b) \( y'' + k \sin y = 0 \), is nonlinear

**Initial and boundary conditions**

Consider the linear D.E.
\[
 y'' + a_1(x)y' + a_2(x) y = g(x) \quad , \quad x \in I
\]

and suppose that \( y(x_0), y'(x_0) \) are given at some point \( x_0 \) in \( I \). The conditions of \( y \) and its derivative at \( x_0 \), are called initial conditions. The problem of solving the D.E. subject to the initial conditions is an initial value problem.

Notation: I.V.P.

Consider the linear D.E.
\[
 y'' + a_1(x)y' + a_2(x) y = 0 \quad , \quad a < x < b
\]

with the boundary conditions:
\[
 \begin{align*}
 \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\
 \beta_1 y(b) + \beta_2 y'(b) &= 0
\end{align*}
\]
The problem of solving the D.E. subject to the boundary conditions is called a boundary value problem.

Notation: B.V.P.

Examples:

a) Consider the D.E.

\[ y'' = -g, \quad \text{where} \quad \frac{d}{dt} \]

and \( g \) is the constant of gravity.

By integrating twice, we get the general solution

\[ y(t) = -\frac{1}{2} gt^2 + c_1 t + c_2, \]

where \( c_1 \) and \( c_2 \) are constants of integration.

Suppose we are given the initial position and the initial velocity: \( y(0) = y_0 \) and \( y'(0) = \dot{y}_0 \), which are the initial conditions, then it is possible to obtain the unique solution of this I.V.P., which is simply

\[ y(t) = -\frac{1}{2} gt^2 + \dot{y}_0 t + y_0. \]

b) Consider the B.V.P.

\[ y'' + k^2 y = 0, \quad 0 < x < 1 \]

with the B.C.: \( y(0) = 0 = y(1) \)

The general solution is \( y(x) = c_1 \cos kx + c_2 \sin kx \)

where \( c_1 \) and \( c_2 \) are arbitrary constants.

Applying the B.C., we get:

\[ 0 = y(0) = c_1, \]

\[ 0 = y(1) \Rightarrow \sin k = 0 \Rightarrow k = n\pi \]

In order to have nontrivial solutions we must have \( k^2 = (n\pi)^2, \quad n = 1, 2, \ldots \), and each \( y_n(x) = \sin n\pi x \) is a solution.
1.1 First Order Linear O.D.E.

**Homogeneous Equations**

**Theorem 1**: Every solution of the D.E.

\[ y' + ay = 0 \]

where \( a \) is a constant

has the form \( y = c e^{-ax} \), where \( c \) is an arbitrary constant.

\[
\begin{align*}
0 &= e^{ax} (y' + ay) = (e^{ax} y)' \\
\Rightarrow e^{ax} y &= \text{constant} = c \\
\Rightarrow y &= ce^{-ax}
\end{align*}
\]

**Note**: The unique solution of the I.V.P.

\[
\begin{cases}
y' + ay = 0 \\
y(x_0) = y_0
\end{cases}
\]

is given by the formula \( y = y_0 e^{-a(x-x_0)} \).

Put \( x = x_0 \) in the general solution \( y = ce^{-ax} \),

we get \( y_0 = y(x_0) = c e^{-ax_0} \) \( \Rightarrow c = y_0 e^{ax_0} \).

By substitution we obtain \( y = y_0 e^{-a(x-x_0)} \).

**Theorem 2**: Consider the D.E.

\[ y' + a(x)y = 0 \]

where \( a(x) \) is continuous on \( I \)

and let \( A(x) = \int a(x) \, dx \).

Then every solution has the form

\[ y = ce^{-A(x)} \]

where \( c \) is an arbitrary constant.

\[
\begin{align*}
0 &= e^{A(x)} (y' + a(x)y) = (e^{A(x)} y)' \\
\Rightarrow e^{A(x)} y &= \text{constant} = c \\
\Rightarrow y &= ce^{-A(x)}
\end{align*}
\]

**Note**: The unique solution of the I.V.P.

\[
\begin{cases}
y' + a(x)y = 0 \\
y(x_0) = y_0
\end{cases}
\]
is given by the formula:
\[ y = y_0 e^{-A(x)} \text{ where } A(x) = \int_{x_0}^{x} a(t) \, dt \]

**Example 1:** 
\[ x^2 y' + 2y = 0, \quad x > 0 \]
\[ y' + \frac{2}{x} y = 0 \]
\[ y = ce^{-\int \frac{2}{x} \, dx} = ce^{-2 \ln x} = cx^{-2} \]

**Example 2:** 
\[ y' + (x - x^{-1}) y = 0, \quad x > 0 \]
\[ y = ce^{-\int (x - x^{-1}) \, dx} = ce^{-x^2/2} \]

**Example 3:** 
\[ y' - (\tan x) y = 0, \quad 0 \leq x < \frac{\pi}{2} \]
\[ y(x) = 1 \]
The general solution is 
\[ y = ce^{\int \tan x \, dx} = \frac{c}{\cos x} \]

Applying the I.C. 
\[ \therefore y = \frac{1}{\cos x} \]

2. **Nonhomogeneous Equations**

**Thm. 3:** Suppose \( a \) and \( g \) are continuous on \( I \), and let \( A(x) = \int a(t) \, dt \).

Then every solution of the D.E.
\[ y' + a(x) y = g(x) \]
has the form: 
\[ y = ce^{-A(x)} + e^{-A(x)} \int e^{A(x)} g(x) \, dx \]
where \( c \) is an arbitrary constant.

\[ \begin{align*}
\int e^{A(x)} (y' + a(x) y) &= e^{A(x)} g(x) \\
\Rightarrow \quad (y e^{A(x)})' &= e^{A(x)} g(x) \\
\Rightarrow \quad y e^{A(x)} &= c + \int e^{A(x)} g(x) \, dx \\
\Rightarrow \quad y &= ce^{-A(x)} + \int e^{A(x)} g(x) \, dx
\end{align*} \]
Note: The unique solution of the I.V.P.
\[ y' + a(\xi)y = g(\xi), \quad y(\xi_0) = y_0. \]
is given by the formula:
\[ y = y_0 e^{-\int_\xi_0^\xi a(\eta)d\eta} + e^{-\int_\xi_0^\xi a(\eta)d\eta} \int_\xi_0^\xi e^{\int_\eta^\xi a(\eta)d\eta} g(\eta)d\eta, \]
where \( A(\xi) = \int_\xi^\infty a(\eta)d\eta \).

Example 4:
The electric current in a circuit is governed by the D.E.
(Kirchhoff law): \( L \frac{di}{dt} + R i = E \),
where \( L, R, E \) are taken here to be constants \( > 0 \).

\[ \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \]

\[ i = c e^{-\frac{R}{L}t} + \frac{E}{R} \int_0^t e^{\frac{R}{L}t} \frac{E}{L} dt = c e^{-\frac{R}{L}t} + \frac{E}{R} \]

As \( t \to \infty \), \( i \to \frac{E}{R} \).

Examples 5:
\[ \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \cos(wt) \]

\[ \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \int_0^t e^{\frac{R}{L}t} \frac{E}{L} \cos(wt) dt \]

\[ = c e^{-\frac{R}{L}t} + \frac{E}{L} \int_0^t e^{\frac{R}{L}t} \frac{E}{L} \cos(wt) dt \]

\[ = c e^{-\frac{R}{L}t} + \frac{E}{L} \left( \frac{R}{L} \cos(wt + \omega \sin wt) \right) \]

\[ = c e^{-\frac{R}{L}t} + \frac{E}{L} \left( \frac{R}{L} \cos(\omega t - \theta) \right), \]

where \( \tan \theta = \frac{wL}{R} \).
Example 6

Newton's law of cooling

\[
\frac{dT}{dt} = k \left( T - T_m \right)
\]

where \( T_m = 70^\circ \) is the medium temperature.

A hot cake, just out of the oven, is at \( 330^\circ \)

(i.e. at \( t = 0 \), \( T = 330 \))

After 3 minutes, \( T = 200^\circ \)

Find \( T \) for all \( t \). When are we able to eat our cake?

The general solution is: \( T - T_m = ce^{kt} \)

\[
T = T_m + ce^{kt}
\]

From the I.C. \( 330 = 70 + c \) \( \Rightarrow c = 260 \)

\[
\therefore T = 70 + 260e^{kt}
\]

We can get the value of \( k \) from the second information: at \( t = 3 \), \( T = 200 \)

i.e. \( 200 = 70 + 260e^{3k} \) \( \Rightarrow \frac{130}{260} = e^{3k} \)

\[
\Rightarrow k = -\frac{1}{3} \ln \frac{130}{260}
\]

\[
\therefore T = 70 + 260e^{-t/3}
\]

at \( t = 9 \) minutes \( T = 70 + 32.5 = 102.5 \)
Note:

As in integration the method of substitution is one of the most important methods of solving O.D.E. Here is a special class of O.D.E. called Bernoulli equation which can be put in linear form by substitution.

Thm: Consider Bernoulli O.D.E.

\[ y' + p(x)y = q(x)y^n \]

where \( n \neq 0, 1 \).

By the substitution \( u = y^{1-n} \), we obtain a linear O.D.E. for \( u \)

\[ u' + (1-n)p = (1-n)q \]

which we can solve by the formula given in this section.

Example 1: \( y' + y = xy^2 \)

\( y(0) = 1 \), \( x > 0 \)

\( u = y^{1-n} \)

\[ u' - u = -x \]

\( u(x) = 1 \)

\( u = ce^x - e^x \int e^{-x} dx = ce^x + x + 1 \)

\( 1 = u(0) = c + 1 \Rightarrow c = 0 \Rightarrow u = x + 1 \)

\( y = \frac{1}{1 + x} \)

Example 2: \( y' + xy = xy^3 \)

\( y(0) = \frac{\sqrt{2}}{2} \), \( y > 0 \)

\( u = y^{1-n} \)

\[ u' - xu = -x \]

\( u(0) = 2 \)

\( u = ce^{x^2/2} \) \( \text{at } x = 0 \)

\( 2 = u(0) = c + 1 \Rightarrow c = 1 \)

\( u = e^x + 1 \Rightarrow y = (e^x + 1)^{-1/2} \)
1.2 The method of separation of variables

Thm.

Consider the D.E.

\[ V(y) \frac{dy}{dx} = u(x) \quad (1) \]

where \( u \) and \( v \) are continuous on:

\[ I : a < x < b \quad \text{and} \quad J : c < y < d \quad (\text{resp.}) \]

and let \( U' = u \) and \( V' = v \).

If the equation

\[ V(y) = U(x) + c \quad (2) \quad c = \text{constant} \]

defines a continuously differentiable function \( y \) on \( I \), then \( y \) is a solution of (1).

Conversely (1) implies (2).

\[
(2) \Rightarrow (1)
\]

\[
\frac{d}{dx} \left[ V(y) - U(x) + c \right]
\]

\[
v(y) y'(x) = u(x)
\]

(1) \Rightarrow (2)

Suppose \( v(y) \frac{dy}{dx} = u(x) \)

integrating from \( x_0 \) to \( x \):

\[
\int_{x_0}^{x} v(y(t)) y'(t) dt = \int_{x_0}^{x} u(t) dt
\]

\[
\int_{y_0}^{y} v(y) dy = \int_{x_0}^{x} u(t) dt
\]

Hence \( V(y) = U(x) + c \)
Example 1
\[ y' = y^2 \quad , \quad y(0) = y_0 > 0 \]
\[
\frac{dy}{y^2} = dx \quad \Rightarrow \quad -\frac{1}{y} = x + c
\]
\[ y = \frac{1}{x+c} \]
I.C. \[ y_0 = y(0) = -\frac{1}{c} \quad \Rightarrow \quad c = -\frac{1}{y_0} \]
\[ y = \frac{1}{x+y_0^{-1}} = \frac{y_0}{1-xy_0} \]

Example 2
\[ (1+x) \frac{dy}{dx} = y \quad , \quad y(0) = 1, \quad x, y > 0 \]
\[
\frac{dy}{y} = \frac{dx}{1+x}
\]
\[ \ln y = \ln (1+x) + c \quad \Rightarrow \quad y(0) = 1 \Rightarrow c = 0 \]
\[ y = 1+x \]

Example 3
\[ y' = ay(1-y) \quad , \quad a > 0, \quad 0 < y < 1 \]
\[ y(0) = y_0 > 0 \]
epidemics equation
\[
\frac{dy}{y(1-y)} = adx
\]
\[
\frac{1}{y} + \frac{1}{1-y} \frac{dy}{dx} = adx
\]
\[ \ln \frac{y}{1-y} = ax + c \quad \Rightarrow \quad \frac{y}{1-y} = Ke^{ax} \]
I.C. \[ \frac{y}{1-y} = K \]
\[ y_0 \]
\[ \frac{y}{1-y} = \frac{y_0 e^{ax}}{1-y_0} \quad \Rightarrow \quad (1+E)\frac{y}{y_0} = E \Rightarrow y = \frac{E}{1+E} \]
\[ y = \frac{y_0 (1-y_0)^{-1} e^{ax}}{1 + y_0 (1-y_0)^{-1} e^{ax}} = \frac{y_0}{y_0 + (1-y_0) e^{ax}} \]
Note: Euler method for solving the D.E.

\[
\frac{dy}{dx} = F\left(\frac{y}{x}\right)
\]

Using the transformation \( u = \frac{y}{x} \)

\[ y = xu \Rightarrow \frac{dy}{dx} = u \, dx + x \, du \]

From the D.E.

\[ u \, dx + x \, du = F(u) \, dx \Rightarrow \]

\[ (F(u) - u) \, dx = x \, du \Rightarrow \]

\[ \frac{dx}{x} = \frac{du}{F(u) - u} \]

which can be solved by the method of separation of variables.

Example

\[
\frac{dy}{dx} = \frac{y^2 + 3xy + x^2}{x^2} = \left(\frac{y}{x}\right)^2 + 3 \left(\frac{y}{x}\right) + 1 = F(u)
\]

where \( F(u) = u^2 + 3u + 1 \), \( u = \frac{y}{x} \)

\[ F(u) - u = u^2 + 2u + 1 = (u + 1)^2 \]

\[ \frac{dx}{x} = \frac{du}{F(u) - u} = \frac{du}{(u + 1)^2} \]

\[ \ln |x| = -\frac{1}{u + 1} + C = -\frac{x}{u + 1} + C \]

\[ \ln |x| + \frac{x}{x + y} = C \]
I.3 Existence and Uniqueness of Solution

Given the I.V.P.
\[ y' = f(x, y) \quad (1) \]
\[ y(x_0) = y_0. \]

how can we be sure if there is a solution? and if there is a solution, is it the only one?

These 2 properties of existence and uniqueness of solution are some of the basic characteristics of physical and real problems.

The answer to these questions are given by the following result, which is probably the most important theorem on O.D.E.

Thm: If \( f \) and \( \frac{\partial f}{\partial y} \) are continuous on the rectangle \( R : \]
\[ (x-x_0, 1 \leq a, \quad 1/2 - a, 1/2 \leq b \]
then there is a unique solution for \((1)\), defined on some interval \( I : 1x-x_1, 1x \leq h \) contained in \( 1x-x, 1x \).

Note: We shall make use of this result during this course, but its proof belongs to a higher level course.

Example 1: \( y' + a(x)y = g(x), \quad y(x_0) = y_0. \)

We have already seen that the formula:
\[ y(x) = y_0 e^{-\int_{x_0}^{x} a(t) \, dt} - e^{-\int_{x_0}^{x} a(t) \, dt} \int_{x_0}^{x} g(t) \, dt, \quad A(x) = \int_{x_0}^{x} a(t) \, dt \]
is the unique solution, if \( a \) and \( g \) are continuous on \( I \).

In this case \( f(x, y) = -a(x)y + g(x) \)
\[ \text{and} \quad \frac{\partial f}{\partial y} = -a(x). \]
Example 2

\[ y' + \frac{2}{x} y = 4x, \quad x > 0, \quad y(1) = 2 \]

\[ y = x^2 + \frac{1}{x^2}, \quad x > 0 \]

\[ a(x) = \frac{2}{x} \]

is not continuous at \( x = 0 \)

Notice that \( x^{-2} \to \infty \) as \( x \to 0 \), for the solution.

Example 3

\[ y' = y^2, \quad y(0) = 1 \]

\[ y = \frac{1}{1-x} \]

\[ f(x, y) = y^2 \]

\[ \frac{\partial f}{\partial y} = 2y \]

Example 4

\[ y' = 3y^{2/3}, \quad y(0) = 0 \]

\[ \frac{dy}{3y^{2/3}} = dx \]

\[ y^{1/3} + c = x \quad \Rightarrow \quad y^{1/3} = x - c \]

\[ 2y^{-1/3} \]

For each \( n = 0, 1, 2, \ldots \), define

\[ \phi_n(x) = \begin{cases} 0, & -\infty < x < n \\ (x-n)^3, & x \geq n \end{cases} \]

Every \( \phi_n \) satisfies the D.E. and the I.C.

We have infinitely many solutions

Note that the condition for \( \frac{\partial f}{\partial y} = 2y^{-1/3} \)

does not satisfy at \( y = 0 \)
I.4 Exact Equations

Def: Exact Equation

A D.E. of the form

\[ M(x,y) \, dx + N(x,y) \, dy = 0 \]  \hspace{1cm} (1)

in a rectangle \( R : a \leq x \leq b, \ c \leq y \leq d \)

is said to be \underline{exact} if there is a function \( F \),

having continuous first partial derivatives, and such that

\[ \frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N \quad \text{in} \ R \]

Note: In this case, the D.E. can be written in the form:

\[ dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy = M \, dx + N \, dy = 0 \]

\[ \Rightarrow F = \text{constant} \]

Example

Let \( F(x,y) = x^2 + y^2 \)

then \( dF = F_x \, dx + F_y \, dy = 2x \, dx + 2y \, dy \)

Therefore \( x \, dx + y \, dy = 0 \) is exact

and the equation is solved by the family of curves

\( x^2 + y^2 = c \)

(Concentric circles centered at the origin)

Remark: Suppose that the equation \( F(x,y) = c \) \hspace{1cm} (2)

defines implicitly a continuously differentiable function \( y \) of \( x \) on an interval \( I \), then by differentiation with respect to \( x \), we get

\[ 0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = M + Ny' \]

\[ \therefore (2) \Rightarrow (1) \]
Thm: Condition for exactness

Suppose $M, N$ and its first partial derivatives are continuous on $R$.

$$M \, dx + N \, dy = 0 \text{ is exact } \iff M_y = N_x$$

$\implies$ Suppose the ODE is exact. Then there is a function $F$ such that $F_x = M$ and $F_y = N$.

Since $M_y = F_{x \gamma}$ and $N_x = F_{y \gamma}$, $M_y = N_x$.

$\iff$ Suppose $M_y = N_x$. We want to show that (1) is exact.

Let $F(x, y) = \int_{x_0}^{x} M(\xi, y) \, d\xi + \int_{y_0}^{y} N(x, \eta) \, d\eta$

We have $F_x = M$

Also $F_y = \int_{x_0}^{x} M_y(\xi, y) \, d\xi + N(x, y)

= \int_{x_0}^{x} N_x(\xi, y) \, d\xi + N(x, y)

= N(x, y)$

$\therefore$ (1) is exact.

Example 1:

$$dx \gamma \, dx + (x^2 + 3y^2) \, dy = 0$$

$M_y = 2x$

$N_x = 2x$ \implies The ODE is exact

$$F(x, y) = \int M \, dx + g(y) = x^2 \gamma + g(y)$$

$$x^2 + 3y^2 = F = x^2 + g'(y) \implies g'(y) = 3y^2 \implies g(y) = y^3$$

$\therefore F(x, y) = x^2 \gamma + y^3 = c$

Example 2:

$$\left( x^2 + y^2 \right) \, dx - \frac{x \gamma}{y^2} \, dy = 0$$

$M_y = -y^2 \neq N_x \Rightarrow$ the ODE is exact

$$F(x, y) = \int M \, dx + g(y) = lnx + \frac{x \gamma}{y} + g(y) \implies g'(y) = 0$$

$\therefore$ $lnx + xy^2 = c$
Example 3: \((\sin x \cos x - xy^2) \, dx + y(1-x^2) \, dy = 0\)

\[ M_y = -2xy = N_x \Rightarrow \text{The D.E. is exact} \]

\[ F(x,y) = \int M \, dx + g(y) \]
\[ = \frac{1}{2} \sin x - \frac{1}{2} x^2 y^2 + g(y) \]
\[ y(1-x^2) = F_y = -x^2 y + g'(y) \Rightarrow g'(y) = y \Rightarrow g(y) = \frac{y^2}{2} \]

\[ \frac{1}{2} \sin^2 x - \frac{1}{2} x^2 y^2 + \frac{y^2}{2} = C \]

or \[ \sin^2 x + y^2(1-x^2) = K \]

Def: Integrating Factors

Suppose \(M \, dx + N \, dy = 0 \quad (1)\) is not exact.

If \(\mu M \, dx + \mu N \, dy = 0 \quad (2)\) is exact
then \(\mu\) is called an integrating factor for \((1)\).

We will suppose that \(M, N\) and \(\mu\) are of class \(C^1\).

Notation: Integrating factor = I.F.

Example 1
\[ (1 + \frac{x}{y}) \, dx - \frac{x^2}{y^2} \, dy = 0 \]

\[ M_y = -\frac{x}{y^2} \]
\[ N_x = 2 \frac{x^2}{y^3} \]

\(\Rightarrow\) the D.E. is not exact

Let us try to see if \(\mu = x^{-1}\) is an integrating factor
the new D.E. is \((x^{-1} + y^{-1}) \, dx - \frac{x}{y^2} \, dy = 0\)

\[ M_y = -\frac{x^{-2}}{y^2} = N_x \Rightarrow \mu = x^{-1} \text{ is indeed an I.F.} \]
Remark: If we can get an integrating factor, then it is possible to solve a non-exact D.E. !

Question:
How do we find an I.F. ?

We try to answer this question by using the condition for exactness.

Here is a simple way:

Case 1: If \( \mu \) is a function of \( x \) only, then
\[
\mu' = \frac{M_y - N_x}{N} \mu = 0
\]

Case 2: If \( \mu \) is a function of \( y \) only, then
\[
\mu' + \frac{M_y - N_x}{M} \mu = 0
\]

\[\begin{array}{l}
\text{Case 1: } (\mu M)_y = (\mu N)_x \\
\quad \Rightarrow \mu M_y = \mu' N + \mu N_x \Rightarrow \mu' \frac{M_y - N_x}{N} \mu = 0
\end{array}\]

\[\begin{array}{l}
\text{Case 2: } (\mu M)_y = (\mu N)_x \\
\quad \mu' M + \mu N_y = \mu N_x \Rightarrow \mu' + \frac{M_y - N_x}{M} \mu = 0
\end{array}\]

Example 2: \((x+1)e^y dx + xe^y dy = 0\) \hspace{1cm} (1)

\[\begin{array}{l}
M_y = (x+1)e^y, \quad N_x = e^y \quad \Rightarrow \text{(1) is not exact}
\end{array}\]

Since \(\frac{M_y - N_x}{N} = 1\), we try case 1:
\[
\mu' - \mu = 0 \quad \Rightarrow \mu = e^x
\]

The new D.E. is \((x+1)e^x dx + xe^{x+y} dy = 0\)

and is exact.
Example 3: \[ y \cos x \, dx + (y + 2) \sin x \, dy = 0 \] (1)

\[ M_y = \cos x \]
\[ N_x = (y + 2) \cos x \]
\[ \oint (1) \text{ is not exact} \]

\[ \frac{My - Nx}{M} = \frac{-y + 1}{y} = -(1 + y^{-1}) \]

This leads to try case 2: \( \mu' = -(1 + y^{-1}) \mu = 0 \)

\[ \mu = e^{\int (1 + y^{-1}) \, dy} = ye^y \]

The new I.E is

\[ y^2 \cos x \, e^y \, dx + (y^2 + 2y) e^y \sin x \, dy = 0 \]

which is exact.

Can we solve this equation?

\[ F(x, y) = \int M \, dx + g(y) = y^2 e^y \sin x + g(y) \]

\[ \sin (y^2 + 2y) e^y = F_y = (y^2 + 2y) e^y \sin x + g'(y) \Rightarrow g'(y) = 0 \]

\[ F(x, y) = y^2 e^y \sin x = \text{constant} \]