MATH-4600  Advanced Calculus
Exam#1  October 5, 2000

Name ________________________________

Instructions.
Please work all five problems.

Please show all work clearly and in reasonable detail. Answers without appropriate supporting work or requested explanations may not receive full credit.

One two-sided sheet of notes is allowed. Use of other notes, books or calculators is not permitted.
1. (a) (10pts) Compute a unit normal vector to the sphere

\[(x - 1)^2 + (y - 2)^2 + z^2 = 3\]

at the point \(P = (0, 1, -1)\).

(b) (5pts) Find an equation for the tangent plane to the sphere at \(P\).
2. (15pts) Consider the function

\[ f(x, y) = \frac{y^2}{(1 - x)^2 + y^2} \]

Where can the function not be continuous? Explain why not.
3. Let $S$ be the surface described by

$$F(x, y, z) = z^3 + 4z - x^2 + xy^2 + 8y - 7 = 0.$$ 

(a) (5pts) Calculate $\partial F/\partial z$ to show that $S$ may be described near all the points where it is defined as a function $z = g(x, y)$. What theorem does this illustrate?

(b) (15pts) Show that there is one critical point of the function $z = g(x, y)$, and find it.
4. Let $\mathbf{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\mathbf{A} = (A_1, A_2, A_3)$ be constant vectors, $\omega$ a constant scalar, and $\mathbf{r} = (x, y, z)$. Suppose that

$$\mathbf{v} = \mathbf{A} \sin(\mathbf{\alpha} \cdot \mathbf{r} - \omega t).$$

The vector $\mathbf{v}$ is called a **vector plane wave**.

(a) (5pts) Show that

$$\frac{\partial \mathbf{v}}{\partial t} = -\omega \mathbf{A} \cos(\mathbf{\alpha} \cdot \mathbf{r} - \omega t).$$

(b) (10pts) Show that

$$\nabla \cdot \mathbf{v} = \mathbf{\alpha} \cdot \mathbf{A} \cos(\mathbf{\alpha} \cdot \mathbf{r} - \omega t).$$
(c) (15 pts) Show that

\[ \nabla \times \mathbf{v} = \mathbf{\alpha} \times \mathbf{A} \cos(\mathbf{\alpha} \cdot \mathbf{r} - \omega t). \]
5. (20pts) The basic equations relating spherical and rectangular coordinates are

\[
\begin{align*}
x &= \rho \sin \varphi \cos \theta \\
y &= \rho \sin \varphi \sin \theta \\
z &= \rho \cos \varphi
\end{align*}
\]

Consider the transformation \( w = (f \circ x)(x, y, z) := g(\rho, \varphi, \theta) \), where the components of \( x \) are given above, and \( f \) and \( g \) are scalar valued differentiable functions. Find the matrices such that

\[
Dg(\rho, \varphi, \theta) = Df(x, y, z)Dx(\rho, \varphi, \theta).
\]