ADVANCED CALCULUS  MATH-4600
Assignment #3  Due: October 18, 2001

1. Given that the function \( z = f(x, y) \) is defined implicitly by the relation
\[ z^3 + 4z - x^2 + xy^2 + 8y - 7 = 0, \]
find and test, all of the critical points of \( f \).
(Hint: Use implicit differentiation to compute \( f_x \) and \( f_y \). Then compute
the second partial derivatives.)

2. (a) By matrix theory, it is known that for any \((n \times n)\) matrix \( A \), the trace
of \( A \) \( (\text{tr}(A) = \sum_{i=1}^{n} a_{ii}, \) which is the sum of its diagonal elements), \( \text{tr}(A) = \sum_{i=1}^{n} \lambda_i, \)
the sum of the eigenvalues, and \( \det(A) = \prod_{i=1}^{n} \lambda_i, \) the product of the
eigenvalues. Use this information and Theorem 2.3 of chapter 4 to show
that if \( f(x) \) is a harmonic function, that is
\[ f_{x_1x_1} + f_{x_2x_2} + \cdots + f_{x_nx_n} = 0, \]
and if \( \det Hf(x_0) \neq 0 \) at a critical point \( x_0 \), then this critical point must
be a saddle point. (Actually, it is also true if \( \det Hf(x_0) = 0 \) when the
function is harmonic, but you don’t have to prove it.)
(b) Locate the critical points and illustrate the result of part (a) by the
function
\[ f(x, y, z) = 2z^3 - 3x^2z - 3y^2z - 2z^2 + x^2 + y^2. \]
Note: It is acceptable to use Maple for the calculations.

3. (a) Text, p.297, #18. Method: Show that the total time
\[ T(\theta_1, \theta_2) = \frac{a}{v_1 \cos \theta_1} + \frac{b}{v_2 \cos \theta_2}. \]
Notes: Observe that, if \( v_1 < v_2, \) in the end there will be incidence angles \( \theta_1, \)
so that there will be no real refraction angle \( \theta_2. \) Such a situation is referred
to as total reflection, which was considered in class. A generalization of
this result to refraction of two media separated by a curve \( h(x, y) = 0 \) is
outlined in a problem in the reference Advanced Calculus, by Amazigo &
Rubenfeld on p.152.
(b) Text: p.294, #12

(over)
4. (a) Text: p. 293, #7.
   (b) Consider the system of ODE’s, stating Newton’s second law with \( m = 1, \)
   \[
   r''(t) = F(r(t)), \quad (\dagger)
   \]
   where \( F \) is the field in part (a), with \( r(t) = (x(t), y(t)) \). Expand this system to 4 equations in four variables say \( x(t) = (x_1(t), x_2(t), x_3(t), x_4(t)) \), where \( x_1(t) = x(t), x_2(t) = x'(t), x_3(t) = y(t), \) and \( x_4(t) = y'(t) \). From (\dagger), now define the \((4 \times 4)\) linear system
   \[
   x' = G(x) := Ax - b.
   \]
   Verify that the critical point at \( c = (-\frac{1}{4}, 0, -\frac{1}{4}, 0) \) is stable by determining the eigenvalues of the matrix \( A = DG(c) \). This shows that the notion of stability of physical equilibria agrees with the mathematical definition which we encountered in differential equations.

5. (a) Text: p.375, #23.
   (b) In a simplified model of a hurricane, the velocity is taken to be purely in the circumferential direction and of magnitude
   \[
   v(r, \theta, z) = \Omega r e^{-\frac{z}{\Lambda}},
   \]
   where \( r, \theta \) and \( z \) are cylindrical coordinates measured from the eye of the hurricane at sea level, with \( \Omega, a \) and \( h \) constants. If the density \( \rho \) of the atmosphere is \( \rho(r, \theta, z) = \rho_0 e^{-\frac{z}{h}}, \) for \( \rho_0 \) constant, find the total kinetic energy
   \[
   E = \frac{1}{2} \iint_W \rho v^2 dV,
   \]
   where \( W \) is the region \( z \geq 0 \), and locate where the velocity has its maximum value. \textbf{Note:} It is acceptable to use Maple for the calculations.