50. In Exercise 40, the calculations \((AB)u\) and \(A(Bu)\) produce the same result. Which calculation requires fewer multiplications of individual matrix entries? (For example, it takes two multiplications to get the \((1, 1)\) entry of \(AB\).

51. The next section will show that all the following calculations produce the same result:

\[ C[A(Bu)] = (CA)(Bu) = [C(AB)]u = C[(AB)u]. \]

Convince yourself that the first expression requires the fewest individual multiplications. [Hint: Forming \(Bu\) takes four multiplications, and thus \(A(Bu)\) takes eight multiplications, and so on.] Count the number of multiplications required for each of the four preceding calculations.

52. Refer to the matrices and vectors in Eq. (11).

\[ a) \text{ Identify the column vectors in } A = [A_1, A_2] \]
\[ \text{ and } D = [D_1, D_2, D_3, D_4]. \]

\[ b) \text{ In part (a), is } A_1 \text{ in } R^2, R^3, \text{ or } R^4? \text{ Is } D_1 \text{ in } R^2, R^3, \text{ or } R^4? \]

\[ c) \text{ Form the } (2 \times 2) \text{ matrix with columns } 
\[ \begin{bmatrix} 4A_1B_1, AB_2 \end{bmatrix}, \text{ and verify that this matrix is the product } AB. \]

\[ d) \text{ Verify that the vector } Dw \text{ is the same as } 
2D_1 + 3D_2 + D_3 + D_4. \]

53. Determine whether the following matrix products are defined. When the product is defined, give the size of the product.

\[ a) \text{ } AB \text{ and } BA, \text{ where } A \text{ is } (2 \times 3) \text{ and } B \text{ is } (3 \times 4) \]
\[ b) \text{ } AB \text{ and } BA, \text{ where } A \text{ is } (2 \times 3) \text{ and } B \text{ is } (2 \times 4) \]
\[ c) \text{ } AB \text{ and } BA, \text{ where } A \text{ is } (3 \times 7) \text{ and } B \text{ is } (6 \times 3) \]
\[ d) \text{ } AB \text{ and } BA, \text{ where } A \text{ is } (2 \times 3) \text{ and } B \text{ is } (3 \times 2) \]
\[ e) \text{ } AB \text{ and } BA, \text{ where } A \text{ is } (3 \times 3) \text{ and } B \text{ is } (3 \times 1) \]

\[ f) A(BC) \text{ and } (AB)C, \text{ where } A \text{ is } (2 \times 3), B \text{ is } (3 \times 5), \text{ and } C \text{ is } (5 \times 4) \]

\[ g) AB \text{ and } BA, \text{ where } A \text{ is } (4 \times 1) \text{ and } B \text{ is } (1 \times 4) \]

54. What is the size of the product \((AB)(CD)\), where \( A \) is \((2 \times 3)\), \( B \) is \((3 \times 4)\), \( C \) is \((4 \times 4)\), and \( D \) is \((4 \times 2)\)? Also calculate the size of \( A[BC]D \) and \( [(AB)C]D \).

55. If \( A \) is a matrix, what should the symbol \( A^2 \) mean? What restrictions on \( A \) are required in order that \( A^2 \) be defined?

56. Set

\[ O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \]
\[ A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } \]
\[ B = \begin{bmatrix} 1 & b \\ b^{-1} & 1 \end{bmatrix}, \]

where \( b \neq 0 \). Show that \( O, A, \) and \( B \) are solutions to the matrix equation \( X^2 - 2X = O \). Conclude that this quadratic equation has infinitely many solutions.

57. Two newspapers compete for subscriptions in a region with 300,000 households. Assume that no household subscribes to both newspapers and that the following table gives the probabilities that a household will change its subscription status during the year.
EXERCISES

The matrices and vectors listed in Eq. (3) are used in several of the exercises that follow.

\[
A = \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 1 \\ 7 & 4 & 3 \\ 6 & 0 & 1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 2 & 1 & 4 & 0 \\ 6 & 1 & 3 & 5 \\ 2 & 4 & 2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix},
\]

\[
E = \begin{bmatrix} 3 & 6 \\ 2 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\]

\[
u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}
\]  

(3)

Exercises 1–25 refer to the matrices and vectors in Eq. (3). In Exercises 1–6, perform the multiplications to verify the given equality or nonequality.

1. \((DE)F = D(EF)\)  
2. \((FE)D = F(ED)\)  
3. \(DE \neq ED\)  
4. \(EF \neq FE\)  
5. \(Fu = Fv\)  
6. \(3Fu = 7Fv\)

In Exercises 7–12, find the matrices.

7. \(A^T\)  
8. \(D^T\)  
9. \(E^T\)  
10. \(A^TC\)  
11. \((Fv)^T\)  
12. \((EF)v\)

In Exercises 13–25, calculate the scalars.

13. \(u^Tv\)  
14. \(v^TFu\)  
15. \(v^TDv\)  
16. \(v^TFv\)  
17. \(u^Tu\)  
18. \(v^Tv\)  
19. \(\|u\|\)  
20. \(\|Dv\|\)  
21. \(\|Au\|\)  
22. \(\|u - v\|\)  
23. \(\|Fv\|\)  
24. \(\|Fv\|\)

25. \(\|(D - E)u\|\)

26. Let \(A\) and \(B\) be \((2 \times 2)\) matrices. Prove or find a counterexample for this statement: \((A - B)(A + B) = A^2 - B^2\).

27. Let \(A\) and \(B\) be \((2 \times 2)\) matrices such that \(A^2 = AB\) and \(A \neq O\). Can we assert that, by cancellation, \(A = B\)? Explain.

28. Let \(A\) and \(B\) be as in Exercise 27. Find the flaw in the following proof that \(A = B\).

Since \(A^2 = AB\), \(A^2 - AB = O\). Factoring yields \(A(A - B) = O\). Since \(A \neq O\), it follows that \(A - B = O\). Therefore, \(A = B\).

29. Two of the six matrices listed in Eq. (3) are symmetric. Identify these matrices.

30. Find \((2 \times 2)\) matrices \(A\) and \(B\) such that \(A\) and \(B\) are symmetric, but \(AB\) is not symmetric. \([Hint: (AB)^T = B^TA^T = BA.]\)

31. Let \(A\) and \(B\) be \((n \times n)\) symmetric matrices. Give a necessary and sufficient condition for \(AB\) to be symmetric. \([Hint: Recall Exercise 30.]\)

32. Let \(G\) be the \((2 \times 2)\) matrix that follows, and consider any vector \(x\) in \(R^2\) where both entries are not simultaneously zero:

\[
G = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad |x_1| + |x_2| > 0.
\]

Show that \(x^T G x > 0\). \([Hint: Write \(x^T G x\) as a sum of squares.]\)

33. Repeat Exercise 32 using the matrix \(D\) in Eq. (3) in place of \(G\).

34. For \(F\) in Eq. (3), show that \(x^T F x \geq 0\) for all \(x\) in \(R^2\). Classify those vectors \(x\) such that \(x^T F x = 0\).

If \(x\) and \(y\) are vectors in \(R^n\), then the product \(x^T y\) is often called an inner product. Similarly, the product \(x y^T\) is often called an outer product. Exercises 35–40 concern outer products; the matrices and vectors are given in Eq. (3). In Exercises 35–40, form the outer products.

35. \(uv^T\)  
36. \(u(Fu)^T\)  
37. \(v(Ev)^T\)

38. \(u(Ev)^T\)  
39. \((Au)(Av)^T\)  
40. \((Av)(Au)^T\)

41. Let \(a\) and \(b\) be given by

\[
a = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.
\]

a) Find \(x\) in \(R^2\) that satisfies both \(x^T a = 6\) and \(x^T b = 2\).

b) Find \(x\) in \(R^2\) that satisfies both \(x^T (a + b) = 12\) and \(x^T a = 2\).

42. Let \(A\) be a \((2 \times 2)\) matrix, and let \(B\) and \(C\) be given by

\[
B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.
\]

a) If \(A^T + B = C\), what is \(A\)?
b) If \( A^T B = C \), what is \( A \) and \( C \)?

Calculate \( BC_1, B_1^T C, (BC_1)^T C_2, \) and \( \|CB_2\| \).

43. Let
\[
A = \begin{bmatrix}
4 & -2 & 2 \\
2 & 4 & -4 \\
1 & 1 & 0
\end{bmatrix}
\]
and \( u = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \).

a) Verify that \( Au = 2u \).

b) Without forming \( A^2 \), calculate the vector \( A^5 u \).

c) Give a formula for \( A^n u \), where \( n \) is a positive integer. What property from Theorem 8 is required to derive the formula?

44. Let \( A, B, \) and \( C \) be \((m \times n)\) matrices such that \( A + C = B + C \). The following statements are the steps in a proof that \( A = B \). Using Theorem 7, provide justification for each of the assertions.

a) There exists an \((m \times n)\) matrix \( O \) such that \( A = A + O \).

b) There exists an \((m \times n)\) matrix \( D \) such that \( A = A + (C + D) \).

c) \( A = (A + C) + D = (B + C) + D \).

d) \( A = B + (C + D) \).

e) \( A = B + O \).

f) \( A = B \).

45. Let \( A, B, C, \) and \( D \) be matrices such that \( AB = D \) and \( AC = D \). The following statements are steps in a proof that if \( r \) and \( s \) are scalars, then \( (rB + sC) = (r + s)D \). Use Theorems 8 and 9 to provide reasons for each of the steps.

a) \( A(rB + sC) = A(rB) + A(sC) \).

b) \( A(rB + sC) = r(AB) + s(AC) = rD + sD \).

c) \( A(rB + sC) = (r + s)D \).

46. Let \( x \) and \( y \) be vectors in \( R^n \) such that \( \|x\| = \|y\| = 1 \) and \( x^T y = 0 \). Use Eq. (1) to show that \( \|x - y\| = \sqrt{2} \).

47. Use Theorem 10 to show that \( A + A^T \) is symmetric for any square matrix \( A \).

48. Let \( A \) be the \((2 \times 2)\) matrix
\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}
\]
Choose some vector \( b \) in \( R^2 \) such that the equation \( Ax = b \) is inconsistent. Verify that the associated equation \( A^T x = A^T b \) is consistent for your choice of \( b \). Let \( x^* \) be a solution to \( A^T x = A^T b \), and select some vectors \( x \) at random from \( R^2 \). Verify that \( \|Ax^* - b\| \leq \|Ax - b\| \) for any of these random choices for \( x \). (In Chapter 3, we will show that \( A^T x = A^T b \) is always consistent for any \((m \times n)\) matrix \( A \) regardless of whether \( Ax = b \) is consistent or not. We also show that any solution \( x^* \) of \( A^T x = A^T b \) satisfies \( \|Ax^* - b\| \leq \|Ax - b\| \) for all \( x \) in \( R^n \); that is, such a vector \( x^* \) minimizes the length of the residual vector \( r = Ax - b \).)

49. Use Theorem 10 to prove each of the following:

a) If \( Q \) is any \((m \times n)\) matrix, then \( Q^T Q \) and \( QQ^T \) are symmetric.

b) If \( A, B, \) and \( C \) are matrices such that the product \( ABC \) is defined, then \( (ABC)^T = C^TB^TA^T \). [Hint: Set \( BC = D \).]

Note: These proofs can be done quickly without considering the entries in the matrices.

50. Let \( Q \) be an \((m \times n)\) matrix and \( x \) any vector in \( R^n \). Prove that \( x^T Q^T Q x \geq 0 \). [Hint: Observe that \( Qx \) is a vector in \( R^m \).]

51. Prove properties 2, 3, and 4 of Theorem 7.

52. Prove property 1 of Theorem 8. [Note: This is a long exercise, but the proof is similar to the proof of part 2 of Theorem 10.]

53. Prove properties 2 and 3 of Theorem 8.

54. Prove properties 2, 3, and 4 of Theorem 9.

55. Prove properties 1 and 3 of Theorem 10.

In Exercises 56–61, determine \( n \) and \( m \) so that \( I_n A = A \) and \( AI_m = A \), where:

56. \( A \) is \((2 \times 3)\) \hspace{1cm} 57. \( A \) is \((5 \times 7)\)

58. \( A \) is \((4 \times 4)\) \hspace{1cm} 59. \( A \) is \((4 \times 6)\)

60. \( A \) is \((4 \times 2)\) \hspace{1cm} 61. \( A \) is \((5 \times 5)\)

62. a) Let \( A \) be an \((n \times n)\) matrix. Use the definition of matrix multiplication to show that \( AI_n = A \) and \( I_n A = A \).

b) Let \( B \) be a \((p \times q)\) matrix. Use the definition of matrix multiplication to show that \( BI_q = B \) and \( I_p B = B \).
In Exercises 16–27, use Definition 12 to determine whether the given matrix is singular or nonsingular. If a matrix $M$ is singular, give all solutions of $M\mathbf{x} = \mathbf{0}$.

16. $A$
17. $B$
18. $C$
19. $AB$
20. $BA$
21. $D$
22. $F$
23. $D + F$
24. $E$
25. $EF$
26. $DE$
27. $F^T$

In Exercises 28–33, determine conditions on the scalars so that the set of vectors is linearly dependent.

28. $v_1 = \begin{bmatrix} 1 \\ a \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

29. $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ a \end{bmatrix}$

30. $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix}$

31. $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ a \\ 3 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 2 \\ b \end{bmatrix}$

32. $v_1 = \begin{bmatrix} a \\ 1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} b \\ 1 \\ c \end{bmatrix}$

33. $v_1 = \begin{bmatrix} 1 \\ a \end{bmatrix}$, $v_2 = \begin{bmatrix} b \\ c \end{bmatrix}$

In Exercises 34–39, the vectors and matrices are from Eq. (10) and Eq. (11). The equations listed in Exercises 34–39 all have the form $M\mathbf{x} = \mathbf{b}$, and all the equations are consistent. In each exercise, solve the equation and express $\mathbf{b}$ as a linear combination of the columns of $M$.

34. $Ax = v_1$
35. $Ax = v_3$
36. $Cx = v_4$
37. $Cx = v_2$
38. $Fx = u_1$
39. $Fx = u_3$

In Exercises 40–45, express the given vector $\mathbf{b}$ as a linear combination of $v_1$ and $v_2$, where $v_1$ and $v_2$ are in Eq. (10).

40. $\mathbf{b} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

41. $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

42. $\mathbf{b} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

43. $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

44. $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

45. $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

In Exercises 46–47, let $S = \{v_1, v_2, v_3\}$.

a) For what value(s) $a$ is the set $S$ linearly dependent?

b) For what value(s) $a$ can $v_3$ be expressed as a linear combination of $v_1$ and $v_2$?

46. $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ a \end{bmatrix}$

47. $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ a \end{bmatrix}$

48. Let $S = \{v_1, v_2, \mathbf{v}_3\}$ be a set of vectors in $\mathbb{R}^3$, where $v_1 = \mathbf{0}$. Show that $S$ is a linearly dependent set of vectors. [Hint: Exhibit a nontrivial solution for either Eq. (5) or Eq. (6)].

49. Let $\{v_1, v_2, \mathbf{v}_3\}$ be a set of nonzero vectors in $\mathbb{R}^m$ such that $v_i^Tv_j = 0$ when $i \neq j$. Show that the set is linearly independent. [Hint: Set $a_1v_1 + a_2v_2 + a_3v_3 = \mathbf{0}$ and consider $\mathbf{0}^Tv\theta$.]

50. If the set $\{v_1, v_2, v_3\}$ of vectors in $\mathbb{R}^m$ is linearly dependent, then argue that the set $\{v_1, v_2, v_3, v_4\}$ is also linearly dependent for every choice of $v_4$ in $\mathbb{R}^m$.

51. Suppose that $\{v_1, v_2, v_3\}$ is a linearly independent subset of $\mathbb{R}^m$. Show that the set $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is also linearly independent.

52. If $A$ and $B$ are $(n \times n)$ matrices such that $A$ is nonsingular and $AB = \mathbf{0}$, then prove that $B = \mathbf{0}$. [Hint: Write $B = [B_1, \ldots, B_n]$ and consider $AB = [AB_1, \ldots, AB_n]$]

53. If $A$, $B$, and $C$ are $(n \times n)$ matrices such that $A$ is nonsingular and $AB = AC$, then prove that $B = C$. [Hint: Consider $A(B - C)$ and use the preceding exercise.]

54. Let $A = [A_1, \ldots, A_{n-1}]$ be an $(n \times (n - 1))$ matrix. Show that $B = [A_1, \ldots, A_{n-1}, Ab]$ is singular for every choice of $\mathbf{b}$ in $\mathbb{R}^{n-1}$.

55. Suppose that $C$ and $B$ are $(2 \times 2)$ matrices and that $B$ is singular. Show that $CB$ is singular. [Hint: By Definition 12, there is a vector $\mathbf{x}_1$ in $\mathbb{R}^2$, $\mathbf{x}_1 \neq \mathbf{0}$, such that $B\mathbf{x}_1 = \mathbf{0}$].

56. Let $\{w_1, w_2\}$ be a linearly independent set of vectors in $\mathbb{R}^2$. Show that if $\mathbf{b}$ is any vector in $\mathbb{R}^2$, then $\mathbf{b}$ is a linear combination of $w_1$ and $w_2$. [Hint: Consider the $(2 \times 2)$ matrix $A = [w_1, w_2]$.]