In Exercises 51–56, determine whether the vector field \( \mathbf{F} \) is conservative. If it is, find a potential function for the vector field.

51. \( \mathbf{F}(x, y, z) = \sin y \mathbf{i} - x \cos y \mathbf{j} + \mathbf{k} \)
52. \( \mathbf{F}(x, y, z) = e^x (y \mathbf{i} + x \mathbf{j}) + \mathbf{k} \)
53. \( \mathbf{F}(x, y, z) = e^x (y \mathbf{i} + x \mathbf{j} + xy \mathbf{k}) \)
54. \( \mathbf{F}(x, y, z) = y^2 z^2 \mathbf{i} + 2xyz^2 \mathbf{j} + 3xy^2 z^2 \mathbf{k} \)
55. \( \mathbf{F}(x, y, z) = \frac{1}{y} \mathbf{i} - \frac{x}{y^2} \mathbf{j} + (2z - 1) \mathbf{k} \)
56. \( \mathbf{F}(x, y, z) = \frac{x}{x^2 + y^2} \mathbf{i} + \frac{y}{x^2 + y^2} \mathbf{j} + \mathbf{k} \)

In Exercises 57–60, find the divergence of the vector field \( \mathbf{F} \).

57. \( \mathbf{F}(x, y, z) = 6x^2 \mathbf{i} - xy^2 \mathbf{j} \)
58. \( \mathbf{F}(x, y, z) = xe^x \mathbf{i} + ye^x \mathbf{j} \)
59. \( \mathbf{F}(x, y, z) = \sin x \mathbf{i} + \cos y \mathbf{j} + z^2 \mathbf{k} \)
60. \( \mathbf{F}(x, y, z) = \ln(x^2 + y^2) \mathbf{i} + xy \mathbf{j} + \ln(y^2 + z^2) \mathbf{k} \)

In Exercises 61–64, find the divergence of the vector field \( \mathbf{F} \) at the indicated point.

<table>
<thead>
<tr>
<th>Vector Field</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{F}(x, y, z) = xyz \mathbf{i} + yz \mathbf{j} + \mathbf{k} )</td>
<td>(1, 2, 1)</td>
</tr>
<tr>
<td>( \mathbf{F}(x, y, z) = x^2 \mathbf{i} - 2xz \mathbf{j} + yz \mathbf{k} )</td>
<td>(2, -1, 3)</td>
</tr>
<tr>
<td>( \mathbf{F}(x, y, z) = e^x \sin y \mathbf{i} - e^x \cos y \mathbf{j} )</td>
<td>(0, 0, 3)</td>
</tr>
<tr>
<td>( \mathbf{F}(x, y, z) = \ln(xyz) (\mathbf{i} + y \mathbf{j} + \mathbf{k}) )</td>
<td>(3, 2, 1)</td>
</tr>
</tbody>
</table>

In Exercises 69 and 70, find \( \text{curl} \ (\mathbf{F} \times \mathbf{G}) \).

69. \( \mathbf{F}(x, y, z) = \mathbf{i} + 2xz + 3yz \)
   \( \mathbf{G}(x, y, z) = x \mathbf{i} - yz \mathbf{j} + \mathbf{k} \)
70. \( \mathbf{F}(x, y, z) = x \mathbf{i} - zy \mathbf{j} + \mathbf{k} \)
   \( \mathbf{G}(x, y, z) = x^2 \mathbf{i} + yz + z^2 \mathbf{k} \)

In Exercises 71 and 72, find \( \text{curl} \ (\text{curl} \mathbf{F}) = \nabla \times ( \nabla \times \mathbf{F}) \).

71. \( \mathbf{F}(x, y, z) = xyz \mathbf{i} + yz \mathbf{j} + \mathbf{k} \)
72. \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} - 2xz \mathbf{j} + yz \mathbf{k} \)

In Exercises 73 and 74, find \( \text{div} \ (\mathbf{F} \times \mathbf{G}) \).

73. \( \mathbf{F}(x, y, z) = \mathbf{i} + 2xz + 3yz \)
   \( \mathbf{G}(x, y, z) = x \mathbf{i} - yz \mathbf{j} + \mathbf{k} \)
74. \( \mathbf{F}(x, y, z) = x \mathbf{i} - zy \mathbf{j} + \mathbf{k} \)
   \( \mathbf{G}(x, y, z) = x^2 \mathbf{i} + yz + z^2 \mathbf{k} \)

In Exercises 75 and 76, find \( \text{div} \ (\text{curl} \mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) \).

75. \( \mathbf{F}(x, y, z) = xy \mathbf{i} + yz + z \mathbf{k} \)
76. \( \mathbf{F}(x, y, z) = x^2 \mathbf{i} - 2xz \mathbf{j} + yz \mathbf{k} \)

In Exercises 77–84, prove the property for vector fields \( \mathbf{F} \) and \( \mathbf{G} \) and scalar function \( f \). (Assume that the required partial derivatives are continuous.)

77. \( \text{curl} \ (\mathbf{F} + \mathbf{G}) = \text{curl} \mathbf{F} + \text{curl} \mathbf{G} \)
78. \( \text{curl} (\nabla f) = \nabla \times (\nabla f) = 0 \)
79. \( \text{div} (\mathbf{F} + \mathbf{G}) = \text{div} \mathbf{F} + \text{div} \mathbf{G} \)
80. \( \text{div} (\mathbf{F} \times \mathbf{G}) = (\text{curl} \mathbf{F}) \cdot (\mathbf{G} - \mathbf{F}) + (\text{curl} \mathbf{G}) \cdot (\mathbf{F} - \mathbf{G}) \)
81. \( \nabla \times (\nabla f + (\nabla \times \mathbf{F})) = \nabla \times (\nabla \times \mathbf{F}) \)
82. \( \nabla \times (\mathbf{F} \cdot \nabla f) = f \nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F} \)
83. \( \text{div} (\mathbf{F} \cdot \nabla f) = f \text{div} \mathbf{F} + \mathbf{F} \cdot \nabla f \)
84. \( \text{div} (\nabla \times \mathbf{F}) = 0 \) (Theorem 14.3)

In Exercises 85–88, let \( \mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \) and let \( f(x, y, z) = \| \mathbf{F}(x, y, z) \| \).

85. Show that \( \nabla (\ln f) = \frac{\mathbf{F}}{f^2} \).
86. Show that \( \nabla \left( \frac{1}{f} \right) = \frac{-\mathbf{F}}{f^2} \).
87. Show that \( \nabla f^n = n f^{n-1} \mathbf{F} \).
88. The Laplacian is the differential operator
   \[ \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]
   and Laplace's equation is
   \[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0. \]
   Any function that satisfies this equation is called harmonic.
   Show that the function \( 1/f \) is harmonic.

89. Wind Speed and Direction  The vector field in the figure gives the upper air wind speeds and directions over the United States on February 24, 2000. If the vectors of greater magnitude depict upper air winds of greater speed, compare the wind speeds and directions over the cities of Phoenix, Arizona and Atlanta, Georgia.
EXERCISES FOR SECTION 14.1

In Exercises 1–6, match the vector field with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a) \[ F(x, y) = x \hat{j} \]
(b) \[ F(x, y) = y \hat{i} \]
(c) \[ F(x, y) = x \hat{i} + 3y \hat{j} \]
(d) \[ F(x, y) = (\frac{1}{x}, \frac{1}{y}) \]
(e) \[ F(x, y) = (x, \sin y) \]
(f) \[ F(x, y) = (\frac{1}{2x}, \frac{1}{2y}) \]

In Exercises 7–16, sketch several representative vectors in the vector field.

7. \[ F(x, y) = \hat{i} + x \hat{j} \]
8. \[ F(x, y) = \frac{2}{y} \hat{i} \]
9. \[ F(x, y) = x \hat{i} + y \hat{j} \]
10. \[ F(x, y) = x \hat{i} - y \hat{j} \]
11. \[ F(x, y, z) = 3 \hat{j} \]
12. \[ F(x, y, z) = x \hat{i} \]
13. \[ F(x, y, z) = 4 \hat{i} + y \hat{j} \]
14. \[ F(x, y, z) = (x^2 + y^2) \hat{i} + j \]
15. \[ F(x, y, z) = i + j + k \]
16. \[ F(x, y, z) = x \hat{i} + y \hat{j} + z \hat{k} \]

In Exercises 17–20, use a computer algebra system to graph several representative vectors in the vector field.

17. \[ F(x, y) = \frac{1}{2}(2x \hat{i} + y \hat{j}) \]
18. \[ F(x, y) = (2y - 3x) \hat{i} + (2y + 3x) \hat{j} \]
19. \[ F(x, y, z) = \frac{\sqrt{x^2 + y^2} + z \hat{k}}{\sqrt{x^2 + y^2} + z^2} \]
20. \[ F(x, y, z) = x \hat{i} - y \hat{j} + z \hat{k} \]

In Exercises 21–26, find the gradient vector field for the scalar function. (That is, find the conservative vector field for the potential function.)

21. \[ f(x, y) = 5x^2 + 3xy + 10y^2 \] 22. \[ f(x, y) = \sin 3x \cos 4y \]
23. \[ f(x, y, z) = z - ye^{i^2} \] 24. \[ f(x, y, z) = \frac{y}{z} - \frac{xz}{y} \]
25. \[ g(x, y, z) = xy \ln(x + y) \] 26. \[ g(x, y, z) = x \arcsin yz \]

In Exercises 27–30, verify that the vector field is conservative.

27. \[ F(x, y) = 12x \hat{i} + 6(x^2 + y) \hat{j} \] 28. \[ F(x, y) = \frac{1}{x^2}(y \hat{i} - x \hat{j}) \]
29. \[ F(x, y) = (\sin y) \hat{i} + x(\cos y) \hat{j} \] 30. \[ F(x, y) = \frac{1}{xy}(y \hat{i} - x \hat{j}) \]

In Exercises 31–34, determine if the vector field is conservative.

31. \[ F(x, y) = 5y^2(3y \hat{i} - x \hat{j}) \]
32. \[ F(x, y) = \frac{1}{\sqrt{x^2 + y^2}}(x \hat{i} + y \hat{j}) \]
33. \[ F(x, y) = \frac{1}{\sqrt[4]{1 - x^2 - y^2}}(y \hat{i} - x \hat{j}) \]
34. \[ F(x, y) = \frac{1}{\sqrt[3]{1 - x^2}}(y \hat{i} - x \hat{j}) \]

In Exercises 35–42, determine whether the vector field is conservative. If it is, find a potential function for the vector field.

35. \[ F(x, y) = 2xy \hat{i} + x^2 \hat{j} \] 36. \[ F(x, y) = \frac{1}{y^2}(y \hat{i} - 2x \hat{j}) \]
37. \[ F(x, y) = x \hat{i} + (2y + x) \hat{j} \] 38. \[ F(x, y) = 3x^2y^2 \hat{i} + 2x^3y \hat{j} \]
39. \[ F(x, y) = \frac{x \hat{i} + y \hat{j}}{x^2 + y^2} \] 40. \[ F(x, y) = \frac{2y}{x} \hat{i} - \frac{x^2}{y^2} \hat{j} \]
41. \[ F(x, y) = e^t(\cos y \hat{i} + \sin y \hat{j}) \] 42. \[ F(x, y) = \frac{2x \hat{i} + 2y \hat{j}}{(x^2 + y^2)^2} \]

In Exercises 43–46, find the curl of \( F \) at the indicated point.

Vector Field \[ \text{Point} \]
43. \[ F(x, y, z) = x \hat{i} + y \hat{j} + z \hat{k} \] (1, 2, 1)
44. \[ F(x, y, z) = x^2 \hat{i} - 2xz \hat{j} + yz \hat{k} \] (2, -1, 3)
45. \[ F(x, y, z) = e^t \sin y \hat{i} - e^t \cos y \hat{j} \] (0, 0, 3)
46. \[ F(x, y, z) = e^{-xz}(i + j + k) \] (3, 2, 0)

In Exercises 47–50, use a computer algebra system to find the curl of the vector field \( F \).

47. \[ F(x, y, z) = \arctan \frac{y}{x} \hat{i} + \ln \sqrt{x^2 + y^2} \hat{j} + k \]
48. \[ F(x, y, z) = \frac{y \hat{i} + x \hat{j}}{y - z} + \frac{y \hat{k}}{z - x} \]
49. \[ F(x, y, z) = \sin(x - y) \hat{i} + \sin(y - z) \hat{j} + \sin(z - x) \hat{k} \]
50. \[ F(x, y, z) = \sqrt{x^2 + y^2 + z^2} \hat{i} + j + k \]
In Exercises 27 and 28, use a computer algebra system to evaluate the integral
\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]
where \( C \) is represented by \( \mathbf{r}(t) \).

27. \( \mathbf{F}(x, y, z) = x^2\mathbf{i} + 6y\mathbf{j} + y^2\mathbf{k} \)
   \( C: \mathbf{r}(t) = ti + t^2\mathbf{j} + \ln rk, \ 1 \leq t \leq 3 \)

28. \( \mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + 2\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} \)
   \( C: \mathbf{r}(t) = ti + tj + e^t\mathbf{k}, \ 0 \leq t \leq 2 \)

Work: In Exercises 29–34, find the work done by the force field \( \mathbf{F} \) on an object moving along the indicated path.

29. \( \mathbf{F}(x, y) = -xi - 2yj \)
   \( C: \ y = x^3 \) from \( (0, 0) \) to \( (2, 8) \)

30. \( \mathbf{F}(x, y) = x^2\mathbf{i} - xyj \)
   \( C: \ x = \cos^3 t, \ y = \sin^2 t \) from \( (1, 0) \) to \( (0, 1) \)

31. \( \mathbf{F}(x, y) = 2xi + yj \)
   \( C: \) counterclockwise around the triangle with vertices \( (0, 0), (1, 0), \) and \( (1, 1) \)

32. \( \mathbf{F}(x, y) = -yi - xj \)
   \( C: \) counterclockwise along the semicircle \( y = \sqrt{4 - x^2} \) from \( (2, 0) \) to \( (-2, 0) \)

33. \( \mathbf{F}(x, y, z) = xi + yj - 5zk \)
   \( C: \mathbf{r}(t) = 2\cos t \mathbf{i} + 2\sin tj + rk, \ 0 \leq t \leq 2\pi \)

34. \( \mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k} \)
   \( C: \) line from \( (0, 0, 0) \) to \( (5, 3, 2) \)

Work: In Exercises 35–36, find the work done by a person weighing 150 pounds walking exactly one revolution up a circular helical staircase of radius 3 feet if the person rises 10 feet.

35. \( (x, y) \) \( \begin{array}{c|cccc}
(0, 0) & \left( \frac{1}{16} \right) & \left( \frac{1}{2}, \frac{1}{2} \right) & \left( \frac{3}{8}, \frac{9}{16} \right) & (1, 1) \\
(5, 0) & (3.5, 1) & (2, 2) & (1.5, 3) & (1, 5)
\end{array} \)

36. \( \mathbf{F}(x, y) \) \( \begin{array}{c|cccc}
(0, 0) & \left( \frac{1}{16} \right) & \left( \frac{1}{2}, \frac{1}{2} \right) & \left( \frac{3}{8}, \frac{9}{16} \right) & (1, 1) \\
(5, 0) & (3.5, 1) & (2, 2) & (1.5, 3) & (1, 5)
\end{array} \)

In Exercises 37 and 38, evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) for each curve. Discuss the orientation of the curve and its effect on the value of the integral.

37. \( \mathbf{F}(x, y) = x^2\mathbf{i} + xyj \)
   (a) \( \mathbf{r}_1(t) = 2ti + (t - 1)j, \ 1 \leq t \leq 3 \)
   (b) \( \mathbf{r}_2(t) = (2(t - 1))i + (2 - t)j, \ 0 \leq t \leq 2 \)

38. \( \mathbf{F}(x, y) = x^2yi + x^3j \)
   (a) \( \mathbf{r}_1(t) = (t + 1)i + t^2j, \ 0 \leq t \leq 2 \)
   (b) \( \mathbf{r}_2(t) = (1 + 2 \cos t)i + (4 \cos^2 t)j, \ 0 \leq t \leq \pi/2 \)

In Exercises 39–42, demonstrate the property that
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \]
regardless of the initial and terminal points of \( C \), if the tangent vector \( \mathbf{r}'(t) \) is orthogonal to the force field \( \mathbf{F} \).

39. \( \mathbf{F}(x, y) = yi - xj \)
   \( C: \mathbf{r}(t) = ti - 2tj \)

40. \( \mathbf{F}(x, y) = -3yi + xj \)
   \( C: \mathbf{r}(t) = ti - t^2j \)

41. \( \mathbf{F}(x, y) = (x^3 - 2x^2)i + \left( x - \frac{y}{2} \right)j \)
   \( C: \mathbf{r}(t) = ti + tj \)

42. \( \mathbf{F}(x, y) = xi + yj \)
   \( C: \mathbf{r}(t) = 3 \sin t i + 3 \cos tj \)

In Exercises 43–46, evaluate the line integral over the path \( C \) given by \( x = 2t, \ y = 10t \), where \( 0 \leq t \leq 1 \).

43. \( \int_C (x + 3y^2) \, dx \)
44. \( \int_C (x + 3y^2) \, dx \)
45. \( \int_C xy \, dx + y \, dy \)
46. \( \int_C (3y - x) \, dx + y^2 \, dy \)

In Exercises 47–54, evaluate the integral
\[ \int_C (2x - y) \, dx + (x + 3y) \, dy \]
along the path.

47. \( C: \) x-axis from \( x = 0 \) to \( x = 5 \)
48. \( C: \) y-axis from \( y = 0 \) to \( y = 2 \)
49. \( C: \) line segments from \( (0, 0) \) to \( (3, 0) \) and \( (3, 0) \) to \( (3, 3) \)
50. \( C: \) line segments from \( (0, 0) \) to \( (0, -3) \) and \( (0, -3) \) to \( (2, -3) \)
51. \( C: \) arc on \( y = 1 - x^2 \) from \( (0, 1) \) to \( (1, 0) \)
52. \( C: \) arc on \( y = x^{3/2} \) from \( (0, 0) \) to \( (4, 8) \)
53. \( C: \) parabolic path \( x = t, \ y = 2t^2 \) from \( (0, 0) \) to \( (2, 8) \)
54. \( C: \) elliptic path \( x = 4 \sin t, \ y = 3 \cos t \) from \( (0, 3) \) to \( (4, 0) \)

Lateral Surface Area: In Exercises 55–62, find the area of the lateral surface (see figure) over the curve \( C \) in the xy-plane and under the surface \( z = f(x, y) \), where

Lateral surface area = \[ \int_C f(x, y) \, ds. \]