3.4 Bases for Subspaces

1. Backsolving the given system yields \( x_1 = x_3 - x_4 \), and \( x_2 = x_4 \). Thus

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
x_3 - x_4 \\
x_4 \\
x_3 \\
x_4 \\
\end{bmatrix} = x_3 \begin{bmatrix}
1 \\
0 \\
1 \\
0 \\
\end{bmatrix} + x_4 \begin{bmatrix}
-1 \\
1 \\
0 \\
1 \\
\end{bmatrix}.
\]

As in Example 5, \( \{[1, 0, 1, 0]^T, [-1, 1, 0, 1]^T\} \) is a basis for \( W \).

2. Backsolving yields \( x_1 = -x_3 - 2x_4 \) and \( x_2 = 2x_3 + x_4 \).
   It follows that \( \{[-1, 2, 1, 0]^T, [-2, 1, 0, 1]^T\} \) is a basis for \( W \).

3. Writing \( x_1 = x_2 - x_3 + 3x_4 \) we have

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
x_2 - x_3 + 3x_4 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = x_2 \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
\end{bmatrix} + x_3 \begin{bmatrix}
-1 \\
0 \\
1 \\
0 \\
\end{bmatrix} + x_4 \begin{bmatrix}
3 \\
0 \\
0 \\
1 \\
\end{bmatrix}.
\]

Thus \( \{[1, 1, 0, 0]^T, [-1, 0, 1, 0]^T, [3, 0, 0, 1]^T\} \) is the desired basis.

4. Writing \( x_1 = x_2 - x_3 \) and noting that \( x_1, x_3 \) and \( x_4 \) are unconstrained variables, we obtain \( \{[1, 1, 0, 0]^T, [-1, 0, 1, 0]^T, [0, 0, 0, 1]^T\} \) as the desired basis.

5. Since \( x_1 = -x_2 \) we have

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
x_2 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = x_2 \begin{bmatrix}
-1 \\
1 \\
0 \\
0 \\
\end{bmatrix} + x_3 \begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
\end{bmatrix} + x_4 \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}.
\]

It follows that \( \{[-1, 1, 0, 0]^T, [0, 0, 1, 0]^T, [0, 0, 0, 1]^T\} \) is a basis for \( W \).

6. Backsolving yields \( x_1 = 2x_4, x_2 = 2x_4, x_3 = x_4 \). Thus \( \{[2, 2, 1, 1]^T\} \) is a basis for \( W \).

7. Backsolving yields \( x_1 = -2x_3 - x_4 \) and \( x_2 = -x_3 \). Thus

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
-x_3 - x_4 \\
x_3 \\
x_3 \\
x_4 \\
\end{bmatrix} = x_3 \begin{bmatrix}
-2 \\
-1 \\
1 \\
0 \\
\end{bmatrix} + x_4 \begin{bmatrix}
0 \\
0 \\
1 \\
1 \\
\end{bmatrix}.
\]

Therefore \( \{[-2, -1, 1, 0]^T, [-1, 0, 0, 1]^T\} \) is a basis for \( W \).
8. Backsolving yields \( x_1 = -x_4 \) and \( x_2 = -x_3 \). Therefore the set 
\( \{ [-1,0,0,1]^T, [0,-1,1,0]^T \} \) is a basis for \( W \).

9. Let \( \{ w_1, w_2 \} \) be the basis found in Exercise 1. (a) \( x = 2w_1 + w_2 \) (b) \( x \) is not in \( W \). (c) \( x = -3w_2 \) (d) \( x = 2w_1 \).

10. Let \( \{ w_1, w_2 \} \) be the basis found in Exercise 2. (a) \( x = w_1 + w_2 \) (b) \( x = 2w_1 - w_2 \) (c) \( x \) is not in \( W \). (d) \( x = -2w_2 \).

11. (a) \( B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \)

(b) Backsolving the reduced system \( Bx = \theta \) yields the solution \( x_1 = -x_3 - x_4, x_2 = -x_3 + x_4 \) for the homogeneous system \( Ax = \theta \). Thus \( x = [x_1, x_2, x_3, x_4]^T \) is in \( N(A) \) if and only if

\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

It follows that \( \{ [-1, -1, 1, 0]^T, [-1, 1, 0, 1]^T \} \) is a basis for \( N(A) \).

(c) It follows from (b) that \( x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = \theta \) if and only if \( x_1 = -x_3 - x_4 \) and \( x_2 = -x_3 + x_4 \). Since \( x_3 \) and \( x_4 \) are unconstrained variables \( \{ A_1, A_2 \} \) is a basis for \( R(A) \). Setting \( x_3 = 1 \) and \( x_4 = 0 \) yields \( x_1 = -1 \) and \( x_2 = -1 \) so \( -A_1 - A_2 + A_3 = \theta \). Therefore \( A_3 = A_1 + A_2 \). Similarly, setting \( x_3 = 0 \) and \( x_4 = 1 \) yields \( A_4 = A_1 - A_2 \).

(d) The nonzero rows of \( B \) form a basis for the row space of \( A \); that is \( \{ [1, 2, 3, -1], [0, -1, -1, 1] \} \) is the desired basis.

12. (a) \( B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \)

(b) The system \( Ax = \theta \) has solution \( x_1 = -x_3 \) and \( x_2 = -x_3 \). Therefore \( \{ [-1, -1, 1]^T \} \) is a basis for \( N(A) \).

(c) \( \{ A_1, A_2 \} \) is a basis for \( R(A) \) and \( A_3 = A_1 + A_2 \).

(d) \( \{ [1, 1, 2], [0, 1, 1] \} \) is a basis for the row space of \( A \).

13. (a) \( B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).
3.4. BASES FOR SUBSPACES

(c) \( \{A_1, A_2\} \) is a basis for \( \mathcal{R}(A) \) and \( A_3 = (3/2)A_1 - A_2 \).

(d) \( \{[2, 1, 2], [0, 1, -1]\} \) is a basis for the row space of \( A \).

17. The matrix \( A^T \) is row equivalent to \( B^T = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). The desired basis is \( \{[1, 3, 1]^T, [0, -1, -1]^T\} \), formed by taking the nonzero columns of \( B \).

18. The matrix \( A^T \) is row equivalent to \( B^T = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \). The desired basis is \( \{[1, 1, 2]^T, [0, 0, 1]^T\} \), formed by taking the nonzero columns of \( B \).

19. The matrix \( A^T \) is row equivalent to \( B^T = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \) so \( \{[1, 2, 2, 0]^T, [0, 1, -2, 1]^T\} \) is a basis for \( \mathcal{R}(A) \).

20. The matrix \( A^T \) is row equivalent to \( B^T = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \) so \( \{[2, 2, 2]^T, [0, -1, 1]^T, [0, 0, 1]^T\} \) is a basis for \( \mathcal{R}(A) \).

21. (a) For the given vectors \( u_1 \) and \( u_2 \), the equation \( x_1 u_1 + x_2 u_2 = \theta \) has solution \( x_1 = -2x_2 \) where \( x_2 \) is an unconstrained variable. Therefore \( \{u_1\} \) is a basis for \( \text{Sp}(S) \), where \( u_1 = [1, 2]^T \).

(b) If \( A = [u_1, u_2] \) then \( A^T \) is row equivalent to \( B^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \). Therefore \( \{[1, 2]^T\} \) is a basis for \( \text{Sp}(S) \).

22. (a) For the given vectors \( u_1, u_2 \) and \( u_3 \), the equation \( x_1 u_1 + x_2 u_2 + x_3 u_3 = \theta \) has solution \( x_1 = (-1/3)x_3 \) and \( x_2 = (-4/3)x_3 \), where \( x_3 \) is arbitrary. Thus \( \{u_1, u_2 \} \) is a basis for \( \text{Sp}(S) \), where \( u_1 = [1, 2]^T \) and \( u_2 = [2, 1]^T \).

(b) If \( A = [u_1, u_2, u_3] \) then \( A^T \) is row equivalent to \( B^T = \begin{bmatrix} 1 & 2 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} \). Therefore \( \{[1, 2]^T, [0, -3]^T\} \) is a basis for \( \text{Sp}(S) \).

23. (a) For the given vectors \( u_1, u_2, u_3, u_4 \), the equation \( x_1 u_1 + x_2 u_2 + x_3 u_3 + x_4 u_4 = \theta \) has solution \( x_1 = -x_3 - 3x_4, x_2 = -x_3 + x_4 \). Since \( x_3 \) and \( x_4 \) are unconstrained variables, \( \{u_1, u_2\} \) is a basis for \( \text{Sp}(S) \), where \( u_1 = [1, 2, 1]^T \) and \( u_2 = [2, 5, 0]^T \).
(b) If \( A = [u_1, u_2, u_3, u_4] \) then \( A^T \) is row equivalent to \( B^T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \).

Therefore \( \{[1, 2, 1]^T, [0, 1, -2]^T\} \) is a basis for \( \text{Sp}(S) \).

24. (a) For the given vectors \( u_1, u_2, u_3, u_4 \), in the equation \( x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4 = \theta \), \( x_4 \) is an unconstrained variable. The desired basis is \( \{u_1, u_2, u_3\} \), where \( u_1 = [1, 2, -1, 3]^T, u_2 = [-2, 1, 2, -1]^T, \) and \( u_3 = [-1, -1, 1, -3]^T \).

(b) If \( A = [u_1, u_2, u_3, u_4] \), then \( A^T \) reduces to \( B^T = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \). Therefore \( \{[1, 2, -1, 3]^T, [0, 1, 0, 0]^T, [0, 0, 0, 5]^T\} \) is a basis for \( \text{Sp}(S) \).

25. (a) Let \( A \) denote the given matrix. The homogeneous system \( Ax = \theta \) has solution \( x_1 = 0, x_2 \) is arbitrary, \( x_3 = 0 \). Thus \( \{[0, 1, 0]^T\} \) is a basis for \( \text{N}(A) \).

(b) Let \( A \) denote the given matrix. The system \( Ax = \theta \) has solution \( x_1 = -x_2 \), where \( x_2 \) and \( x_3 \) are arbitrary. Thus \( \{[-1, 1, 0]^T, [0, 0, 1]^T\} \) is a basis for \( \text{N}(A) \).

(c) The system \( Ax = \theta \) has solution \( x_1 = -x_2, x_3 = 0 \), where \( x_2 \) is arbitrary. The set \( \{[-1, 1, 0]^T\} \) is a basis for \( \text{N}(A) \).

26. (a) \( \{[1, 1]^T, [0, 1]^T\} \). (b) \( \{[1, 1]^T\} \). (c) \( \{[1, 1]^T, [0, 1]^T\} \).

27. The equation \( x_1v_1 + x_2v_2 + x_3v_3 = \theta \) has solution \( x_1 = -2x_3, x_2 = -3x_3, x_3 \) arbitrary. In particular, \( x_1 = -2, x_2 = -3, x_3 = 1 \) is a nontrivial solution and the set \( S \) is linearly dependent. Moreover, from \( -2v_1 - 3v_2 + v_3 = \theta \) we obtain \( v_3 = 2v_1 + 3v_2 \). If \( v \) is in \( \text{Sp}(S) \) then \( v = a_1v_1 + a_2v_2 + a_3v_3 = (a_1 + 2a_3)v_1 + (a_2 + 3a_3)v_2 \) so \( v \) is in \( \text{Sp}\{v_1, v_2\} \). It follows that \( \text{Sp}\{v_1, v_2, v_3\} = \text{Sp}\{v_1, v_2\} \).

28. The subsets \( \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\} \) are bases for \( R^2 \).

29. The subsets are \( \{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \) and \( \{v_1, v_2, v_4\} \). Note that \( v_4 = 3v_2 - v_3 \).

30. By Theorem 12 of Section 1.8, the matrix \( V = [v_1, v_2, v_3] \) is nonsingular. Thus, by Theorem 13 of Section 1.8, the system of equations \( Ax = b \) has a solution for each \( b \) in \( R^3 \); that is each vector \( b \) in \( R^3 \) can be written in the form \( x_1v_1 + x_2v_2 + x_3v_3 = b \). This shows that \( R^3 = \text{Sp}(B) \) so \( B \) is a basis for \( R^3 \).
3.5. DIMENSION

31. Set $V = \{v_1, v_2, v_3\}$. By assumption the system $Ax = b$ has a solution for every $b$ in $R^3$. By Theorem 13 of Section 1.8, $V$ is a nonsingular matrix. Therefore, by Theorem 12 of Section 1.8, the set $\{v_1, v_2, v_3\}$ is linearly independent.

32. The set $S$ is linearly independent so, by Exercise 30, $S$ is a basis for $R^3$.

33. The set $S$ is linearly dependent so $S$ is not a basis for $R^3$.

34. The set $S$ is linearly dependent so $S$ is not a basis for $R^3$.

35. If $u = [u_1, u_2, u_3]^T$ then $u$ is in Sp($S$) if and only if $4u_1 - 2u_2 + u_3 = 0$. In particular, Sp($S$) $\neq R^3$ and $S$ is not a basis for $R^3$.

36. A vector $w = [w_1, w_2, w_3]^T$ is in Sp($\{v_1, v_2\}$) if and only if $w_1 + w_3 = 0$. In particular $w = [0, 0, 1]^T$ is not a linear combination of $v_1$ and $v_2$.

37. (a) By Theorem 11 of Section 1.8, any set of three or more vectors in $R^2$ is linearly dependent and is not a basis for $R^2$.

(b) Suppose $\{v\}$ is a basis for $R^2$. Then $e_1 = a_1v$ and $e_2 = a_2v$ for some nonzero scalars $a_1$ and $a_2$. But then $a_2e_1 - a_1e_2 = 0$, contradicting the fact that $\{e_1, e_2\}$ is a linearly independent set. We conclude that $\{v\}$ is not a basis for $R^2$. It follows that every basis for $R^2$ contains exactly two vectors.

38. If $v^T = [x_1, x_2, \ldots, x_n]$ then the constraints $v^T u_i = 0, 1 \leq i \leq p$, yield a homogeneous system of $p$ equations in the unknowns $x_1, x_2, \ldots, x_n$. By Theorem 4 of Section 1.4 the system has nontrivial solutions.

Suppose $v = a_1 u_1 + a_2 u_2 + \cdots + a_p u_p$. Then $\|v\|^2 = v^T v = v^T (a_1 u_1 + a_2 u_2 + \cdots + a_p u_p) = a_1 v^T u_1 + a_2 v^T u_2 + \cdots + a_p v^T u_p = 0$, contradicting that $v$ is a nonzero vector.

39. By Theorem 11 of Section 1.8, any set of $n + 1$ or more vectors in $R^n$ is linearly dependent so it is not a basis for $R^n$. By Exercise 38, any set of less than $n$ vectors cannot span $R^n$. Therefore a basis for $R^n$ must contain exactly $n$ vectors.

3.5 Dimension

1. $S$ contains only one vector and dim($R^2$) = 2, so by property 2 of Theorem 9, $S$ does not span $R^2$.

2. $S$ does not span $R^2$ by property 2 of Theorem 9

3. Since $S$ contains three vectors and dim($R^2$) = 2, $S$ is linearly dependent by property 1 of Theorem 9.
4. \( S \) is linearly dependent by property 1 of Theorem 9.

5. Since \( u_4 \neq \theta \), \( S \) is a linearly dependent set; for example \( 0u_1 + au_4 = \theta \) for any nonzero scalar \( a \). Also \( S \) does not span \( \mathbb{R}^2 \) since
   \[ \text{Sp}\{u_1, u_4\} = \text{Sp}\{u_1\}. \]

6. \( S \) is linearly dependent since, for example, \( 3u_1 - u_2 = \theta \).

7. \( S \) contains two vectors and \( \dim(\mathbb{R}^3) = 3 \) so by property 2 of Theorem 9, \( S \) does not span \( \mathbb{R}^3 \).

8. \( S \) does not span \( \mathbb{R}^3 \) by property 2 of Theorem 9.

9. Since \( S \) contains four vectors and \( \dim(\mathbb{R}^3) = 3 \), \( S \) is linearly dependent by property 1 of Theorem 9.

10. It is easily checked that \( S \) is a linearly independent set. Therefore, by property 3 of Theorem 9, \( S \) is a basis for \( \mathbb{R}^2 \).

11. It is easily checked that \( S \) is a linearly independent set. Since \( S \) contains two vectors and \( \dim(\mathbb{R}^2) = 2 \) it follows from property 3 of Theorem 9 that \( S \) is a basis for \( \mathbb{R}^2 \).

12. The set \( S \) is linearly independent so, by property 3 of Theorem 9, \( S \) is a basis for \( \mathbb{R}^3 \).

13. It is easily shown by direct calculation that \( S \) is a linearly dependent set. Therefore \( S \) is not a basis for \( \mathbb{R}^3 \).

14. The set \( S \) is linearly independent so, by property 3 of Theorem 9, \( S \) is a basis for \( \mathbb{R}^3 \).

15. If we write \( x_1 = 2x_2 - x_3 + x_4 \) then the procedure described in Example 5 of Section 2.4 yields a basis \( \{[2, 1, 0, 0]^T, [-1, 0, 1, 0]^T, [1, 0, 0, 1]^T\} \) for \( W \). It follows that \( \dim(W) = 3 \).

16. \( \dim(W) = 3 \).

17. Following the procedure used in Example 5 of Section 2.4, we obtain a basis \( \{[1, -1, 0, 0]^T, [2, 0, -1, 0]^T\} \) for \( W \). In particular \( \dim(W) = 2 \).

18. \( \dim(W) = 2 \).

19. The set \( \{[-1, 3, 2, 1]^T\} \) is a basis for \( W \), so \( \dim(W) = 1 \).

20. \( \dim(W) = 1 \).

21. The homogeneous system \( Ax = \theta \) has solution \( x_1 = -2x_2 \).
   Therefore \( \{[-2, 1]^T\} \) is a basis for \( \mathcal{N}(A) \) and \( \text{nullity}(A) = 1 \). Since \( 2 = \text{rank}(A) + \text{nullity}(A) \), it follows that \( \text{rank}(A) = 1 \).
22. The set \( \{[2, 1, 1]^T\} \) is a basis for \( N(A) \). Therefore nullity \( (A) = 1 \) and rank \( (A) = 2 \).

23. The homogeneous system \( Ax = \theta \) has solution \( x_1 = -5x_3, x_2 = -2x_3 \). Thus \( \{[-5, -2, 1]^T\} \) is a basis for \( N(A) \) and nullity \( (A) = 1 \). Since \( 3 = \text{rank} (A) + \text{nullity} (A) \), it follows that rank \( (A) = 2 \).

24. The set \( \{[2, -1, 1, 0]^T\} \) is a basis for \( N(A) \). Therefore nullity \( (A) = 1 \) and rank \( (A) = 3 \).

25. \( A^T \) reduces to \( B^T = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \). It follows that \( \{[1, -1, 1]^T, [0, 2, 3]^T\} \) is a basis for \( R(A) \). Consequently rank \( (A) = 2 \). Since \( 3 = \text{rank} (A) + \text{nullity} (A) \), it follows that nullity \( (A) = 1 \).

26. The matrix \( A^T \) reduces to \( B^T = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). Therefore \( \{[1, 2, 2]^T, [0, 2, -1]^T\} \) is a basis for \( R(A) \), rank \( (A) = 2 \) and nullity \( (A) = 2 \).

27. (a) Following the methods of Example 7 in Section 2.4, let \( A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \end{bmatrix} \). Then \( A^T \) reduces to \( B^T = \begin{bmatrix} 1 & 1 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \). It follows that \( \{[1, 1, -2]^T, [0, -1, 1]^T, [0, 0, 1]^T\} \) is a basis for \( W \). In particular \( \dim(W) = 3 \).

(b) Following the procedure in (a), we obtain a basis \( \{[1, 2, -1, 1]^T, [0, 1, -1, 1]^T, [0, 0, -1, 4]^T\} \) for \( W \). In particular, \( \dim(W) = 3 \).

28. \( W = \{x \in \mathbb{R}^4 : x_1 + 2x_2 - 3x_3 - x_4 = 0 \} \). It follows that \( \dim(W) = 3 \).

29. The constraints \( a^T x = 0, b^T x = 0 \) and \( c^T x = 0 \) yield the homogeneous system of equations \( x_1 - x_2 = 0, x_1 - x_3 = 0, x_2 - x_3 = 0 \). Solving we obtain \( x_1 = x_3 \) and \( x_2 = x_3 \) where \( x_3 \) and \( x_4 \) are arbitrary. Thus \( \{[1, 1, 1, 0]^T, [0, 0, 0, 1]^T\} \) is a basis for \( W \) and \( \dim(W) = 2 \).

30. Following the procedure described in the hint, suppose we have obtained a linearly independent subset \( S_k = \{w_1, \ldots, w_k\} \) of \( W \). If \( S_k \) spans \( W \) we are done. If not there exists a vector \( w_{k+1} \) in \( W \) such that \( w_{k+1} \) is not in \( \text{Sp}(S_k) \). Suppose \( a_1 w_1 + \cdots + a_k w_k + a_{k+1} w_{k+1} = \theta \). Now \( a_{k+1} = 0 \) since otherwise we could solve for \( w_{k+1} \), contradicting that \( w_{k+1} \) is not in \( \text{Sp}(S_k) \). Since \( S_k \) is linearly independent, it follows