This test contains FOUR pages (including this page) and FIVE questions.

Make sure your copy is complete.

Total number of marks : 100

To obtain full marks you should answer all questions.
1. **[15 marks]**

   (a) Derive the normal equations for the least-squares approximation to the data values \((x_i, y_i), \ i = 1, \ldots, n\), of the form
   
   \[ f(x) = a + b \sin x. \]

   (b) Hence determine the least-squares approximation to the data below of the form \(f(x) = a + b \sin x\), i.e. find the values of \(a\) and \(b\).

\[
\begin{array}{c|ccc}
\quad & -\frac{\pi}{2} & 0 & \frac{\pi}{2} \\
\hline
\quad & 2 & 0 & -1 \\
\end{array}
\]

2. **[10 marks]**

   Find an \(O(h^2)\) approximation for \(f'(x)\) that uses \(f(x), f(x-h)\) and \(f(x+2h)\).

3. **[15 marks]**

   Consider the differential equation
   
   \[ y'(t) = g(y). \]

   The trapezoidal method may be used to calculate the solution at time \(t_{i+1}\) from the solution at time \(t_i\). The difference equation has the form
   
   \[ y_{i+1} = y_i + \frac{h}{2}(g(y_i) + g(y_{i+1})), \]

   where \(y_i = y(t_i)\) and \(h = t_{i+1} - t_i\). Is this method A-stable? Give reasons for your answer.
4. [25 marks]

Below is a contour plot for the function

\[ f(x, y) = 0.5x^2 + 2.5y^2. \]

(a) We wish to find the vector \( \mathbf{x}_m = (x_m, y_m) \) that minimizes \( f(x, y) \).

Let \( \mathbf{x}_0 = (5, 1) \) be a first approximation to \( \mathbf{x}_m \). Find a second approximation \( \mathbf{x}_1 \) using each of the following:

i. Newton’s method

ii. the Steepest Descent method.

(b) Consider the point \( \mathbf{x}_1 \) calculated by Newton’s method. What is particularly significant about this point?

(c) Indicate on the contour plot the value of \( \mathbf{x}_0 \), and the value of \( \mathbf{x}_1 \) computed by the Steepest Descent method. In addition, without making any further calculations, indicate on the contour plot the approximate direction of descent that would be taken from \( \mathbf{x}_1 \) in order to calculate the next approximation \( \mathbf{x}_2 \). Give reason(s) why you have chosen this approximate direction of descent.
5. [35 marks]

Consider the initial value problem given by:

\[ y'(t) = -y^3, \quad t > 0 \]

where \( y(0) = 1 \).

(a) Perform one step of the Modified Euler method (RK2) to compute a value of \( y \) at \( t = 0.1 \).

(b) Perform one step of the backward Euler method to compute a value of \( y \) at \( t = 0.1 \).

Note

- Use one iteration of Newton's method to solve the non-linear algebraic equation in part (b).
- Use the value of \( y \) at \( t = 0.1 \) obtained by performing one step of Euler's method as the starting guess for the Newton iteration.

(c) Obtain the percentage relative error in each of your answers to parts (a) and (b), compared with the exact solution

\[ y(t) = \sqrt{1/(2t + 1)}. \]