

## Chapter 5.4: Regular singular points

We now study solutions of

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

near a *regular singular point*.

- Singular point  $x_0$  :  $P(x_0) = 0$ ;
- Regular singular point  $x_0$ :
- $\lim_{x \rightarrow x_0} (x - x_0) \frac{Q(x)}{P(x)}$  finite, i.e.  $\frac{Q(x)}{P(x)}$  is no worse than  $(x - x_0)^{-1}$ ;
- $\lim_{x \rightarrow x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$  finite. i.e.  $\frac{R(x)}{P(x)}$  is no worse than  $(x - x_0)^{-2}$ .

## Bessel's equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0.$$

Thus,  $P(x) = x^2$ ,  $Q(x) = x$ ,  $R(x) = (x^2 - \nu^2)$ .  
There is a singular point at  $x = 0$ . Is it regular?

We note that

$$Q(x)/P(x) = \frac{x}{x^2} = \frac{1}{x}.$$

This does satisfy the first condition to be regular, since the singularity is  $1/x$ . More formally,

$$\lim_{x \rightarrow 0} xQ(x)/P(x) = 1.$$

## Bessel's equation (cont.)

Also,

$$R(x)/P(x) = (x^2 - \nu^2)/x^2.$$

This blows up like  $1/x^2$  at  $x = 0$  as long as  $\nu \neq 0$ . This also satisfies the condition to be regular. More formally,

$$\lim_{x \rightarrow 0} x^2 R(x)/P(x) = -\nu^2 < \infty.$$

Thus,  $x = 0$  is a regular singularity.

## Legendre's equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

There are singular points at  $x = \pm 1$ . Are they regular?

They both are since  $(1 - x^2) = (1 - x)(1 + x)$ . So  $\frac{1}{P} = \frac{1}{(1-x)(1+x)}$  has the singularity  $\frac{1}{x-1}$  at  $x = 1$ , and  $\frac{1}{1+x}$  at  $x = -1$ . It only blows up to order 1. The singular points have to be regular.

## Legendre's equation

Let us show more completely that  $x = 1$  is a regular singular point:

- $$\lim_{x \rightarrow 1} (x-1) \frac{Q(x)}{P(x)} = \lim_{x \rightarrow 1} (x-1) \frac{-2x}{(1-x)(1+x)} = \lim_{x \rightarrow 1} \frac{2x}{(1+x)} = 1.$$
- $$\lim_{x \rightarrow 1} (x-1)^2 \frac{R(x)}{P(x)} = \lim_{x \rightarrow 1} (x-1)^2 \frac{\alpha(\alpha+1)}{(1-x)(1+x)} = \lim_{x \rightarrow 1} (x-1) \frac{\alpha(\alpha+1)}{(1+x)} = 0.$$

## Behavior of solutions at regular singular points

In future chapters, we will consider how solutions behave as  $x \rightarrow x_0$ , a regular singular point. In general, the solutions are not analytic there, but have the form  $(x - x_0)^r f(x)$  where  $r$  is a real number and  $f$  is analytic. So the solutions can be singular at the regular singular point, but in fairly simple ways. We also consider whether infinity is a regular singular point, which controls the behavior of the solutions at infinity.

## An equation with regular singular points and only analytic solutions

Consider the equation:  $x^2y'' - 2xy' + 2y = 0$ .  
It has a regular singular point at  $x = 0$ .

It is easy to check that two independent solutions are  $y_1 = x$ ,  $y_2 = x^2$ . Hence every solution is a polynomial

$$y = c_1x + c_2x^2.$$

So what impact does the singularity at  $x = 0$  have on solutions? The answer is: you cannot pose the initial value problem at  $x = 0$ , i.e. you cannot find a solution with  $y(0) = y_0, y'(0) = y'_0$ . In fact, both  $x, x^2$  are zero at  $x = 0$ , so all solutions must be zero at  $x = 0$ !!

## Irregular singular point

Consider the equation

$$x^2 y'' + y' + y = 0.$$

Then  $x = 0$  is an irregular singular point because

$$\lim_{x \rightarrow 0} x \frac{Q(x)}{P(x)} = \lim_{x \rightarrow 0} x \frac{1}{x^2} = \infty.$$

Otherwise said, if you divide by  $P$ , the coefficient of  $y'$  blows up like  $1/x^2$ , not  $1/x$ , as for a regular singular point.



## Singular point at infinity

These are important because many of the basic equations have singular points at infinity. Whether the singularity is regular or not determines how the solutions grow at infinity.

The definition is as follows.

Let  $\xi = \frac{1}{x} \iff x = \frac{1}{\xi}$ . Write the equation entirely in terms of  $\xi$ . Then the equation has a singular point at infinity if the  $\xi$ -equation has a singular point at  $\xi = 0$ . It is a regular singular point at infinity if the  $\xi$  equation has a regular singular point at  $\xi = 0$ .

## Rewriting the equation

Let  $x = 1/\xi \iff \xi = 1/x$ . Then:

- $\frac{d}{dx} = \frac{d\xi}{dx} \frac{d}{d\xi} = -\frac{1}{x^2} \frac{d}{d\xi} = -\xi^2 \frac{d}{d\xi}$ .
- $\frac{d^2}{dx^2} = [-\xi^2 \frac{d}{d\xi}][-\xi^2 \frac{d}{d\xi}] = \xi^4 \frac{d^2}{d\xi^2} + 2\xi^3 \frac{d}{d\xi}$ .

So the equation  $[P(x)\frac{d^2}{dx^2} + Q(x)\frac{d}{dx} + R]y = 0$  becomes

$$P\left(\frac{1}{\xi}\right)\left[\xi^4 \frac{d^2}{d\xi^2} Y + 2\xi^3 \frac{d}{d\xi} Y\right] + Q\left(\frac{1}{\xi}\right)\left[-\xi^2 \frac{d}{d\xi} Y\right] + R\left(\frac{1}{\xi}\right)Y = 0.$$

## Rewriting the equation cont.

Here,  $Y(\xi) = y(\frac{1}{\xi})$ .

Collecting like terms, this simplifies to:

$$P(\frac{1}{\xi})\xi^4 Y'' + [2\xi^3 P(\frac{1}{\xi}) - \xi^2 Q(\frac{1}{\xi})]Y' + R(\frac{1}{\xi})Y = 0.$$

Then  $\xi = 0$  is an ordinary point if

$$\frac{2\xi^3 P(\frac{1}{\xi}) - \xi^2 Q(\frac{1}{\xi})}{P(\frac{1}{\xi})\xi^4} = \frac{2}{\xi} - \frac{Q(\frac{1}{\xi})}{P(\frac{1}{\xi})\xi^2}$$

and

$$\frac{R(\frac{1}{\xi})}{P(\frac{1}{\xi})\xi^4}$$

are analytic at  $\xi = 0$

## Regular singular at infinity

Also,  $\xi = 0$  is a regular singular point if

- $\xi \cdot \left[ \frac{2}{\xi} - \frac{Q(\frac{1}{\xi})}{P(\frac{1}{\xi})\xi^2} \right]$  has a finite limit at  $\xi = 0$ ,  
and if
- $\xi^2 \frac{R(\frac{1}{\xi})}{P(\frac{1}{\xi})\xi^4}$  has a finite limit at  $\xi = 0$ .

# Legendre

Consider Legendre's equation:

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

If we put  $\xi = 1/x$ , and simplify a bit, we get the equation in  $\xi$ :

$$\xi^2(\xi^2 - 1)Y'' + [2\xi^3]Y' + \alpha(\alpha + 1)Y = 0.$$

The point  $\xi = 0$  is a singular point. So Legendre's equation does have a singular point at infinity.

If we divide by the leading coefficient we get

$$Y'' + \frac{2\xi^3}{\xi^2(\xi^2 - 1)}Y' + \frac{\alpha(\alpha + 1)}{\xi^2(\xi^2 - 1)}Y = 0.$$

The middle term does not blow up at  $\xi = 0$  and the third term blows up like  $1/\xi^2$ . Thus, the point at infinity is a regular singular point.