On the Global Solution of Linear Programs with Linear Complementarity Constraints

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1. Introduction and Motivation
2. MIP formulation of LPCCs
3. Global solution framework
   - Master Problem
   - Cuts for the Master Problem
4. Refinements
   - Cut Sparsification
5. Computational Results
   - Feasible LPCCs
   - Unbounded LPCCs
   - Infeasible LPCCs
   - Box-constrained quadratic programs
6. Conclusions
Abstract

This talk presents a parameter-free integer-programming based algorithm for the global resolution of a linear program with linear complementarity constraints (LPCC). The cornerstone of the algorithm is a minimax integer program formulation that characterizes and provides certificates for the three outcomes—infeasibility, unboundedness, or solvability—of an LPCC. Computational results demonstrate that the algorithm can handle infeasible, unbounded, and solvable LPCCs effectively.
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Standard form LPCC

Let $c \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, $f \in \mathbb{R}^k$, $q \in \mathbb{R}^m$, $A \in \mathbb{R}^{k \times n}$, $B \in \mathbb{R}^{k \times m}$, $M \in \mathbb{R}^{m \times m}$, and $N \in \mathbb{R}^{m \times n}$ be given. Find $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ to globally solve the linear program with complementarity constraints (LPCC):

$$\begin{align*}
\text{minimize} \quad & c^T x + d^T y \\
\text{subject to} \quad & Ax + By \geq f \\
\text{and} \quad & 0 \leq y \perp q + Nx + My \geq 0,
\end{align*}$$

(We write “LPCC” but say “LPEC” because it is easier to pronounce.)
Preliminary observations

An LPCC is equivalent to $2^m$ linear programs, each called a piece and derived from a subset $\alpha \subseteq \{1, \cdots, m\}$ with complement $\bar{\alpha}$:

$$\text{minimize} \quad c^T x + d^T y$$

subject to

$$A x + B y \geq f$$

$$(q + N x + M y)_{\alpha} \geq 0 = y_\alpha$$

and

$$(q + N x + M y)_{\bar{\alpha}} = 0 \leq y_{\bar{\alpha}}$$

Thus, there are 3 states of an LPCC in general:

- **infeasibility**—all pieces are infeasible
- **unboundedness**—one piece is feasible and unbounded below
- **global solvability**—one piece is feasible and all feasible pieces are bounded below.
Preliminary observations

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\[
\text{minimize} \quad \begin{pmatrix} x, y \end{pmatrix}^T \begin{bmatrix} c & d \end{bmatrix} \\
\text{subject to} \quad \begin{bmatrix} A & B \end{bmatrix} \begin{pmatrix} x, y \end{pmatrix} \geq \begin{pmatrix} f \end{pmatrix} \\
( q + Nx + My )_\alpha \geq 0 = y_\alpha \\
\quad \text{and} \quad ( q + Nx + My )_{\bar{\alpha}} = 0 \leq y_{\bar{\alpha}}
\]

Thus, there are 3 states of an LPCC in general:

- **infeasibility**—all pieces are infeasible
- **unboundedness**—one piece is feasible and unbounded below
- **global solvability**—one piece is feasible and all feasible pieces are bounded below.
Goals

To develop a finite-time algorithm to resolve an LPCC in one of its 3 states, without complete enumeration of all the pieces and without any a priori assumptions and/or bounds.

To provide certificates for the respective states at termination:

- an infeasible piece, if LPCC is infeasible
- an unbounded piece, if LPCC is feasible but unbounded below
- a globally optimal solution, if it exists.

To leverage the state-of-the-art advances in linear and integer programming.
Fundamental importance

The LPCC plays the same important role in disjunctive nonlinear programs as a linear program does in convex programs.

Additionally, it has many applications of its own:

**Novel paradigms in mathematical programming**
- hierarchical optimization
- inverse optimization

**Key formulations for**
- B-stationary conditions of MPECs
  - verification and computation without MPEC-constraint qualification
- global resolution of nonconvex quadratic programs
Outline

1. Introduction and Motivation

2. **MIP formulation of LPCCs**

3. Global solution framework
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4. Refinements
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Equivalent Integer Program

Given a sufficiently large parameter $\theta$ and denoting the vector of ones by $\mathbf{1}$, get an equivalent mixed integer problem:

$$\min_{(x,y,z)} c^T x + d^T y$$

subject to

$$Ax + By \geq f$$
$$\theta z \geq q + Nx + My \geq 0$$
$$\theta(\mathbf{1} - z) \geq y \geq 0$$

and

$$z \in \{0, 1\}^m$$
Dual problem for a fixed $z$

Dual DP($\theta; z$):

maximize \quad f^T \lambda + q^T(u^+ - u^-) - \theta \left[ z^T u^+ + (1 - z)^T v \right]

subject to \quad A^T \lambda - N^T(u^+ - u^-) = c

\quad B^T \lambda - M^T(u^+ - u^-) - v \leq d

and \quad (\lambda, u^\pm, v) \geq 0,

whose feasible region, assumed nonempty throughout,

$$\Xi \equiv \left\{ (\lambda, u^\pm, v) : A^T \lambda - N^T(u^+ - u^-) = c, B^T \lambda - M^T(u^+ - u^-) - v \leq d \right\}$$

is independent of $\theta$.

Note: $\Xi \neq \emptyset \iff \exists (\lambda, u)$ with $\lambda \geq 0$ such that $A^T \lambda + N^T u = c$. 
Removing the parameter $\theta$

- Any feasible solution $(x^0, y^0)$ of the LPCC induces a pair $(\theta_0, z^0)$, where $\theta_0 > 0$ and $z^0 \in \{0, 1\}^m$, such that the pair $(x^0, y^0)$ is feasible to the LP$(\theta, z^0)$ for all $\theta \geq \theta_0$, and

$$ (q + Nx^0 + My^0)_i > 0 \quad \Rightarrow \quad z^0_i = 1 $$

$$ (y^0)_i > 0 \quad \Rightarrow \quad z^0_i = 0. $$

- Conversely, if $(x^0, y^0)$ is feasible to the LP$(\theta, z^0)$ for some $\theta \geq 0$, then $(x^0, y^0)$ is feasible to the LPCC.

- If $(x^0, y^0)$ is an optimal solution to the LPCC, then it is optimal to the LP$(\theta, z^0)$ for all pairs $(\theta, z^0)$ such that $\theta \geq \theta_0$ and $(\theta_0, z^0)$ are as specified above; moreover, for each $\theta > \theta_0$, any optimal solution $(\hat{\lambda}, \hat{u}^\pm, \hat{v})$ of the DLP$(\theta, z^0)$ satisfies

$$ (z^0)^T \hat{u}^+ + (1 - z^0)^T \hat{v} = 0 $$
Removing the parameter $\theta$, continued

Thus, the limiting dual problem for large $\theta$ can be expressed:

$$
D(z) 
\begin{align*}
\text{maximize} & \quad f^T \lambda + q^T (u^+ - u^-) \\
\text{subject to} & \quad A^T \lambda - N^T (u^+ - u^-) = c \\
& \quad B^T \lambda - M^T (u^+ - u^-) - v \leq d \\
& \quad z^T u^+ + (1 - z)^T v = 0 \\
\text{and} & \quad (\lambda, u^{\pm}, v) \geq 0,
\end{align*}
$$

If $z_i = 1$ then $u_i^+ = 0$.
If $z_i = 0$ then $v_i = 0$. 
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The Master Problem

In order to find the best choice for $z$ and to resolve LPCC, use a **logical Benders decomposition method** (Hooker; see also Codato and Fischetti).

Initially every binary $z$ is feasible. **Satisfiability cuts** are added to restrict $z$ based on the solution of the dual problem $D(z)$.

The Master Problem is a Satisfiability Problem.
The algorithm

Outline:

1. **Initialize** the Master Problem with all binary $z$ feasible.
2. If the Master Problem is infeasible, **STOP** with determination of the solution of LPCC.
3. Find a feasible $\bar{z}$ for the Master Problem.
4. Solve the subproblem $D(\bar{z})$.
5. If LPCC proven unbounded, **STOP**.
6. Update the Master Problem and return to Step 2.
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6. Update the Master Problem and return to Step 2.
Implications of solving $D(\bar{z})$: $D(\bar{z})$ finite

If $D(\bar{z})$ is feasible with finite optimal value $\phi(\bar{z})$ then this value gives the optimal value on the corresponding piece of LPCC.

Thus, we obtain an upper bound on the optimal value of LPCC and can restrict attention in the Master Problem to better pieces.

If the optimal solution to $D(\bar{z})$ is feasible in $D(z)$ for some other $z$ then the value of LPCC on the piece corresponding to $z$ must also be at least $\phi(\bar{z})$.

So use a point cut to remove all such $z$ from the Master Problem, based on the optimal solution $(\bar{\lambda}, \bar{u}^{\pm}, \bar{v})$ to $D(\bar{z})$:

$$\sum_{i: \bar{u}^+_i > 0} z_i + \sum_{i: \bar{v}_i > 0} (1 - z_i) \geq 1$$

This logical Benders cut will force at least one of these components of $u^+$ or $v$ to be zero in all future subproblems.
Implications of solving $D(\bar{z})$: $D(\bar{z})$ unbounded

If $D(\bar{z})$ is unbounded then the corresponding piece of LPCC is infeasible.

Have a ray for $D(\bar{z})$.

Cut off all $z$ in the Master Problem for which this ray $(\bar{\lambda}, \bar{u}^{\pm}, \bar{v})$ is feasible in $D(z)$, using a ray cut:

$$
\sum_{i: \bar{u}_i^+ > 0} z_i + \sum_{i: \bar{v}_i > 0} (1 - z_i) \geq 1
$$
Implications of solving $D(\bar{z})$: $D(\bar{z})$ infeasible

If $D(\bar{z})$ is infeasible then the corresponding piece of LPCC is either infeasible or unbounded.

Solve a homogenized version of $D(\bar{z})$ to determine the case:

$$\begin{align*}
\text{maximize} & \quad f^T \lambda + q^T (u^+ - u^-) \\
\text{subject to} & \quad A^T \lambda - N^T (u^+ - u^-) = 0 \\
& \quad B^T \lambda - M^T (u^+ - u^-) - v \leq 0 \\
& \quad \bar{z}^T u^+ + (1 - \bar{z})^T v = 0 \\
\text{and} & \quad (\lambda, u^\pm, v) \geq 0,
\end{align*}$$

If $D_0(\bar{z})$ is unbounded then the corresponding primal problem is infeasible. Thus, the corresponding piece of LPCC is infeasible, so we can again add a ray cut.

If $D_0(\bar{z})$ has optimal value 0 then LPCC is unbounded.
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Some useful auxiliary steps

Simple cuts (Audet, Savard, and Zghal; JOTA, in print) to tighten the joint constraints $Ax + By \geq f$ using the complementarity restrictions.

LPCC feasibility recovery to improve LPCC upper bounds if possible.

Cut sparsification to tighten up the cuts added to the Master Problem.
Cut Sparsification

The fewer variables included in a point cut or ray cut, the tighter the satisfiability constraint.

We use various heuristic procedures to try to sparsify the cut. These heuristics require the solution of linear programs.

In the case of a ray cut, we are looking for an irreducible infeasible set (IIS) of constraints for the primal problem $P(\bar{z})$ that is dual to $D(\bar{z})$. 
Initial feasible region. Take $\bar{z} = (1, 0, 0)$. 
Add cut \((1 - z_1) + z_2 + z_3 \geq 1\) to cut off \(\bar{z} = (1, 0, 0)\)
Sparsify to \((1 - z_1) + z_3 \geq 1\), cuts off \(z = (1, 1, 0)\)
Sparsify further to \((1 - z_1) \geq 1\), cuts off 2 more pts
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Feasible LPCCs with $B = 0$, $A \in \mathbb{R}^{200 \times 300}$, and 300 complementarities

<table>
<thead>
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<th>Prob</th>
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Feasible general LPCCs with $B \neq 0$, $A \in \mathbb{R}^{55 \times 50}$, and 50 complementarities.

<table>
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<tr>
<th>Prob</th>
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## Unbounded LPCCs with 50 complementarities

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# iters = number of Master Problem iterations  
# cuts = number of satisfiability constraints in Master Problem at termination  
# LPs = number of LPs solved, excluding the pre-processing step
Infeasible LPCCs with 50 complementarities

<table>
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LPCC formulation of QP

The optimal solution to the **box-constrained quadratic program**

\[
\begin{align*}
\text{min} & \quad c^T x + \frac{1}{2} x^T Q x \\
\text{subject to} & \quad 0 \leq x \leq 1
\end{align*}
\]

can be found by solving the LPCC:

\[
\begin{align*}
\text{min} & \quad c^T x - \mathbf{1}^T y \\
\text{subject to} & \quad 0 \leq x \perp c + Q x + y \geq 0 \\
& \quad 0 \leq \mathbf{1} - x \perp y \geq 0
\end{align*}
\]
Preliminary computational results

Problems typically solved in a few seconds:
\[ n = 50 \text{ with density } 25\% \]
\[ n = 75 \text{ with density } 10\% \]
\[ n = 100 \text{ with density } 5\% \]

Problems where some instances cannot currently be solved:
\[ n = 100 \text{ with density } 10\% \]

Problems where the current implementation typically has great difficulties:
\[ n = 75 \text{ with density } 25\% \]
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   - Feasible LPCCs
   - Unbounded LPCCs
   - Infeasible LPCCs
   - Box-constrained quadratic programs
6. Conclusions
The logical Benders decomposition method can successfully find the global solution to large feasible LPCC instances, often finding better solutions than NLP methods which determine local minimizers.

The method successfully identifies infeasible or unbounded LPCC instances.

The method can be used to solve bounded quadratic programming problems. Extension: We can also formulate a general QP as an LPCC, which can even identify unbounded QPs.