Introduction

The most successful interior point methods work with both primal and dual iterates. The primal iterates \( x \) and the dual slacks \( s \) are both strictly positive at each iteration. The limit points typically satisfy strict complementarity, with either \( x_i \) or \( s_i \) strictly positive.

When the optimal primal solution is unique and nondegenerate, the limit point is the optimal BFS. In general, primal-dual interior point methods typically find solutions that are in the interior of the optimal face. We work with the standard primal-dual pair, with the dual slacks written out explicitly:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax = b \quad (P) \\
\end{align*}
\]

\[
\begin{align*}
\max_{y \in \mathbb{R}^m, s \in \mathbb{R}^n} & \quad b^T y \\
\text{subject to} & \quad A^T y + s = c \quad (D) \\
\end{align*}
\]

The sets of optimal solutions are denoted:

\[
\begin{align*}
\Omega_P := \{ x \in \mathbb{R}^n : x \text{ solves } (P) \} \\
\Omega_D := \{ (y, s) \in \mathbb{R}^m \times \mathbb{R}^n : (y, s) \text{ solves } (D) \}
\end{align*}
\]

The following Lemma follows directly from strong duality:

**Lemma 1.** The sets \( \Omega_P \) and \( \Omega_D \) are either both empty or both nonempty.

**Definition 1.** A strictly feasible point for \( (P) \) is a point \( \bar{x} \in \mathbb{R}^n \) satisfying \( A\bar{x} = b \) and \( \bar{x} > 0 \). A strictly feasible point for \( (D) \) is a point \( (\bar{y}, \bar{s}) \in \mathbb{R}^m \times \mathbb{R}^n \) satisfying \( A^T \bar{y} + \bar{s} = c \) and \( \bar{s} > 0 \).

**Theorem 1** (Wright, Thm 2.3, page 26). Assume \( (P) \) and \( (D) \) are both feasible. The problem \( (D) \) has a strictly feasible solution if and only if \( \Omega_P \) is nonempty and bounded. The problem \( (P) \) has a strictly feasible solution if and only if \( \Omega_s := \{ s : (y, s) \in \Omega_D \} \) is nonempty and bounded.

**Proof.** We prove the first equivalence using a primal-dual pair of LPs, and leave the second equivalence as an exercise. Let \( z^* \) denote the optimal value of \( (P) \) and \( (D) \) and let \( e \) denote the vector of ones. Consider the primal-dual pair:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad -e^T x \\
\text{subject to} & \quad Ax = b \quad (Pz) \\
\end{align*}
\]

\[
\begin{align*}
\max_{y \in \mathbb{R}^m, s \in \mathbb{R}^n, \pi \in \mathbb{R}} & \quad b^T y - z^* \pi \\
\text{subject to} & \quad A^T y - c\pi + s = -e \quad (Dz) \\
\end{align*}
\]

The feasible region of \( (Pz) \) is \( \Omega_P \).
Assume $\Omega_P$ is bounded: Then $(P_Z)$ has a finite optimal value, so $(D_z)$ is feasible, with feasible solution $(\hat{y}, \hat{s}, \hat{\pi})$. Break into cases depending on the value of $\hat{\pi}$.

- If $\hat{\pi} > 0$: Let $\bar{y} = \frac{1}{\hat{\pi}} \hat{y}$. Then $\bar{s} = c - A^T \bar{y} = \frac{1}{\hat{\pi}} (e + \hat{s}) > 0$, so $(D)$ has a strictly feasible solution.

- If $\hat{\pi} = 0$: Let $(\tilde{y}, \tilde{s})$ be feasible in $(D)$. Let $\bar{y} = \tilde{y} + \hat{y}$. Then $\bar{s} = c - A^T \bar{y} = \bar{s} - A^T \hat{y} = \bar{s} + \hat{s} + e > 0$, so $(D)$ has a strictly feasible solution.

Assume $(D)$ has a strictly feasible solution $(\hat{y}, \hat{s})$ so $\hat{s} > 0$: Let $\bar{\pi} = 1/\min \{\hat{s}_i\}$. Let $\bar{y} = \bar{\pi} \hat{y}$. Then $c \bar{\pi} - A^T \bar{y} = \hat{s} \bar{\pi} \geq e$. Rearranging gives $A^T \bar{y} - c \bar{\pi} \leq -e$, so $(D_z)$ is feasible. Thus, $(P_Z)$ has a finite optimal value, so $\Omega_P$ is bounded.

(See the text for an alternative proof of part of the lemma.)

2 A partition of the indices: examples

We define two subsets $B$ and $N$ of the indices $\{1, \ldots, n\}$:

\[
B := \{i \in \{1, \ldots, n\} : x^*_i > 0 \text{ for some } x^* \in \Omega_P\}
\]
\[
N := \{i \in \{1, \ldots, n\} : s^*_i > 0 \text{ for some } (y^*, s^*) \in \Omega_D\}
\]

From complementary slackness, we have $B \cap N = \emptyset$.

**Example 1.** $B$ is the set of basic variables in the (unique) optimal BFS:

\[
\begin{align*}
\min_x & \quad 3x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 = 1 \\
& \quad x_i \geq 0, \; i = 1, 2
\end{align*}
\]
\[
\begin{align*}
\max_{y,s} & \quad y_1 \\
\text{subject to} & \quad y_1 + s_1 = 3 \\
& \quad y_1 + s_2 = 1 \\
& \quad s_i \geq 0, \; i = 1, 2
\end{align*}
\]

The unique optimal solution is $x^* = (0, 1), \; y = 1, \; s = (2, 0)$. We have $B = \{2\}, \; N = \{1\}$.

**Example 2.** $B$ is a subset of any set of basic variables.

\[
\begin{align*}
\min_x & \quad -x_1 + x_2 \\
\text{subject to} & \quad x_1 + x_2 + x_3 = 1 \\
& \quad x_1 + x_4 = 1 \\
& \quad x_i \geq 0, \; i = 1, \ldots, 4
\end{align*}
\]
\[
\begin{align*}
\max_{y,s} & \quad y_1 + y_2 \\
\text{subject to} & \quad y_1 + y_2 + s_1 = -1 \\
& \quad y_1 + s_2 = 1 \\
& \quad y_1 + s_3 = 0 \\
& \quad s_i \geq 0, \; i = 1, \ldots, 4
\end{align*}
\]

Unique optimal primal solution is $x^* = (1, 0, 0, 0)$, so $B = \{1\}$. Optimal dual solutions are $y^* = (-1, 0)$ with $s^* = (0, 2, 1, 0)$, $y^* = (0, -1)$ with $s^* = (0, 1, 0, 1)$, and any convex combination of these two solutions. Thus, $N = \{2, 3, 4\}$. 

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Example 3. $\mathcal{B}$ is a superset of any set of basic variables.

\[
\begin{align*}
\text{min } & \quad x_1 + x_2 \\
\text{subject to } & \quad x_1 + x_2 - x_3 = 1 \\
& \quad x_i \geq 0, \ i = 1, \ldots, 3
\end{align*}
\]

\[
\begin{align*}
\text{max } & \quad y_1 \\
\text{subject to } & \quad y_1 + s_1 = 1 \\
& \quad y_1 + s_2 = 1 \\
& \quad -y_1 + s_3 = 0 \\
& \quad s_i \geq 0, \ i = 1, 2, 3
\end{align*}
\]

Optimal primal solutions are all points of the form $x^* = (t, 1 - t, 0)$ for $0 \leq t \leq 1$, so $\mathcal{B} = \{1, 2\}$. Unique optimal dual solution is $y_1^* = 1$, $s^* = (0, 0, 1)$, so $\mathcal{N} = \{3\}$.

Example 4. The problem has multiple primal and multiple dual optimal solutions.

\[
\begin{align*}
\text{min}_x & \quad x_1 + x_2 + x_3 \\
\text{subject to } & \quad 2x_1 + 3x_2 + x_3 - x_4 = 4 \\
& \quad 2x_1 + x_2 + 3x_3 - x_5 = 4 \\
& \quad x_1 + x_2 + x_3 - x_6 = 2 \\
& \quad x_1 \geq 0, \ i = 1, \ldots, 6
\end{align*}
\]

$\mathcal{B} = \{1, 2, 3\}$. These three columns are linearly dependent.

Dual problem:

\[
\begin{align*}
\text{min}_{y,s} & \quad 4y_1 + 4y_2 + 2y_3 \\
\text{subject to } & \quad 2y_1 + 2y_2 + y_3 + s_1 = 1 \\
& \quad 3y_1 + y_2 + y_3 + s_2 = 1 \\
& \quad y_1 + 3y_2 + y_3 + s_3 = 1 \\
& \quad -y_1 + s_4 = 0 \\
& \quad -y_2 + s_5 = 0 \\
& \quad -y_3 + s_6 = 0 \\
& \quad s_i \geq 0, \ i = 1, \ldots, 6
\end{align*}
\]

$\mathcal{N} = \{4, 5, 6\}$.

Note that in each example we have $\mathcal{B} \cup \mathcal{N} = \{1, \ldots, n\}$. For each problem, there is a pair of optimal solutions with $x^* + s^* > 0$: take the averages of the extreme point solutions.
3 A partition of the indices: theorem

**Theorem 2** (Goldman-Tucker Strict Complementarity). For any linear program with \( \Omega_P \) and \( \Omega_D \) nonempty, it holds that \( \mathcal{B} \cup \mathcal{N} = \{1, \ldots, n\} \). There exists an optimal solution with \( x^* + s^* > 0 \).

*Proof.* We prove this by contradiction, so we assume \( J := \{1, \ldots, n\} \setminus (\mathcal{B} \cup \mathcal{N}) \) is nonempty. We use a theorem of the alternative. We denote the \( k \)th column of \( A \) by \( A_k \) and we let \( A_B \) denote the columns of \( A \) corresponding to \( \mathcal{B} \). Pick \( i \in J \) and set up the two systems

\[
\begin{align*}
A_i^T w &< 0 \\
-A_j^T w &\geq 0 \quad j \in J \setminus i \quad (I) \\
A_B^T z & = A_i \\
\mu &\geq 0 \\
z &\text{ free}
\end{align*}
\]

Exactly one of these two systems has a solution. The proof is very similar to Farkas and is left as an exercise.

We show that if \( (I) \) is consistent then \( i \in \mathcal{N} \), and if \( (II) \) is consistent then \( i \in \mathcal{B} \). In both cases, this is a contradiction.

**Assume (I) holds:**

Let \((y^*, s^*)\) be a point in \( \Omega_D \) with \( s^*_k > 0 \ \forall k \in \mathcal{N} \). Let \( \bar{w} \) be a solution to \( (I) \).

Note that \( y^* + \epsilon w \) is feasible, at least for small positive values of \( \epsilon \), since \( s(\epsilon) = c - A^T(y^* + \epsilon w) \geq 0 \). Further, \( s_k(\epsilon) = 0 \) for \( k \in \mathcal{B} \), so \( s(\epsilon)^T x^* = 0 \) for any primal optimal \( x^* \in \Omega_P \).

In addition, \( s_i(\epsilon) > 0 \), so by the definition of \( \mathcal{N} \), we have \( i \in \mathcal{N} \). This is a contradiction.

**Assume (II) holds:**

Let \( \bar{\mu}, \bar{z} \) satisfy \( (II) \). Let \( x^* \in \Omega_P \). The following point is feasible in \( (P) \) for small positive \( \zeta \):

\[
x_k(\zeta) = \begin{cases} 
  x_k^* - \zeta \bar{z}_k & \text{if } k \in \mathcal{B} \\
  \zeta & \text{if } i = k \\
  \zeta \bar{\mu}_k & \text{if } k \in J \setminus i \\
  0 & \text{if } k \in \mathcal{N}
\end{cases}
\]

Since \( s_B = 0 \) and \( s_j = 0 \ \forall j \in J \) for any \((y^*, s^*) \in \Omega_D\), we have \( x(\zeta)^T s^* = 0 \), so \( x(\zeta) \) is optimal for \( (P) \). Since \( x_i(\zeta) > 0 \), we have that \( i \in \mathcal{B} \), a contradiction.

Thus, we get a contradiction in either case, so we must have \( J = \emptyset \) and \( \mathcal{B} \cup \mathcal{N} = \{1, \ldots, n\} \). \( \square \)