A Simplex Iteration

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Consider the linear programming problem

$$\begin{align*}
\text{min} & \quad 8x_1 + 4x_2 + 5x_3 + 6x_4 \\
\text{subject to} & \quad x_1 + 2x_2 - x_3 - x_4 = 1 \\
& \quad -x_1 - 5x_2 + 2x_3 + 3x_4 = 1 \\
& \quad x_i \geq 0 \; \forall i
\end{align*}$$

We have

$$A = \begin{bmatrix}
1 & 2 & -1 & -1 \\
-1 & -5 & 2 & 3
\end{bmatrix}, \quad b = \begin{bmatrix}
1 \\
1
\end{bmatrix}, \quad \text{and} \quad c^T = [8 \ 4 \ 5 \ 6].$$

Columns 1 and 3 of $A$ are linearly independent, so we initialize with $B$ equal to these two columns:

$$B = \begin{bmatrix}
1 & -1 \\
-1 & 2
\end{bmatrix},$$

and then

$$N = \begin{bmatrix}
2 & -1 \\
-5 & 3
\end{bmatrix}, \quad c_B = \begin{bmatrix}
8 \\
5
\end{bmatrix}, \quad c_N = \begin{bmatrix}
4 \\
6
\end{bmatrix}, \quad x_B = \begin{bmatrix}
x_1 \\
x_3
\end{bmatrix}, \quad x_N = \begin{bmatrix}
x_2 \\
x_4
\end{bmatrix}. $$

The corresponding basic solution is obtained by calculating $B^{-1}b$:

$$x_B = \begin{bmatrix}
x_1 \\
x_3
\end{bmatrix} = B^{-1}b = \begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
1
\end{bmatrix} = \begin{bmatrix}
3 \\
2
\end{bmatrix}. $$

Thus, we have a basic feasible solution $x^1 = [3, \ 0, \ 2, \ 0]^T$. The objective function value is

$$c^T x^1 = c_B^T B^{-1} b = 34.$$

We next express the LP in the form

$$\begin{align*}
\text{min} & \quad c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N \\
\text{subject to} & \quad x_B + B^{-1} N x_N = B^{-1} b \\
& \quad x_B, x_N \geq 0
\end{align*}$$

This requires the following calculations:

$$B^{-1} N = \begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
2 & -1 \\
-5 & 3
\end{bmatrix} = \begin{bmatrix}
-1 & 1 \\
-3 & 2
\end{bmatrix}.$$ 

and

$$c_N^T - c_B^T B^{-1} N = [4, \ 6] - [8, \ 5] \begin{bmatrix}
-1 & 1 \\
-3 & 2
\end{bmatrix} = [4, \ 6] - [-23, \ 18] = [27, \ -12].$$
(This vector is the vector of reduced costs.) Thus we can express our original linear programming problem equivalently as

\[
\begin{align*}
\min & \quad 34 + 27x_2 - 12x_4 \\
\text{subject to} & \quad x_1 - x_2 + x_4 = 3 \\
& \quad x_3 - 3x_2 + 2x_4 = 2 \\
& \quad x_i \geq 0 \forall i.
\end{align*}
\]

(Notice the order of the variables.) Since the cost of \(x_4\) is negative in this formulation and \(x_4\) is currently zero, we can try to improve the solution by increasing \(x_4\). The first constraint tells us that we have (with \(x_2 = 0\))

\[x_1 = 3 - x_4,
\]

so we can’t make \(x_4\) any bigger than 3 without violating the nonnegativity restriction on \(x_1\). Similarly, the second constraint tells us that we have

\[x_3 = 2 - 2x_4,
\]

so we can’t make \(x_4\) any bigger than 1 without violating the nonnegativity restriction on \(x_3\). (This examination to find the largest possible increase in the variable entering the basis is known as the minimum ratio test.) Thus, we set \(x_4\) to 1 and remove \(x_3\) from the basis, giving a new basis of \(x_1\) and \(x_4\). (This is a pivot from one basic feasible solution to another.)

We can now repeat the whole process over again from the new basic feasible solution. The matrix \(B\) now consists of columns 1 and 4 of \(A\):

\[
B = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix},
\]

We have

\[
N = \begin{bmatrix} 2 & -1 \\ -5 & 2 \end{bmatrix}, \quad c_B = \begin{bmatrix} 8 \\ 6 \end{bmatrix}, \quad c_N = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \quad x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}, \quad x_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}.
\]

The corresponding basic solution is obtained by calculating \(B^{-1}b\):

\[
x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = B^{-1}b = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.
\]

Thus, we have the basic feasible solution \(x^2 = [2, 0, 0, 1]^T\). The objective function value is

\[
c^T x^2 = c_B^T B^{-1} b = 22,
\]

which is indeed an improvement on the previous objective function value of 34. We next express the LP in the form

\[
\min \quad c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N \\
\text{subject to} \quad x_B + B^{-1} N x_N = B^{-1} b \quad x_B, x_N \geq 0
\]
This requires the following calculations:

\[
B^{-1}N = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -5 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}
\]

and

\[
c_N^T - c_B^T B^{-1} N = [4, 5] - \frac{1}{2} [8, 6] \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} = [4, 5] - [4, 5] = [9, 6].
\]

Thus, we have another equivalent formulation of our original linear programming problem:

\[
\begin{align*}
\text{min} & \quad 22 + 9x_2 + 6x_3 \\
\text{subject to} & \quad x_1 + 0.5x_2 - 0.5x_3 = 2 \\
& \quad x_4 - 1.5x_2 + 0.5x_3 = 1 \\
& \quad x_i \geq 0 \quad \forall i.
\end{align*}
\]

Since the reduced costs are all nonnegative, this solution is actually optimal.

**Note:** Let \( z \) be the objective function, so \(-z + 8x_1 + 4x_2 + 5x_3 + 6x_4 = 0\). Then the pivot we performed can be calculated exactly like a pivot in Gaussian elimination, by pivoting on \( x_4 \) in the second constraint of the formulation:

\[
\begin{align*}
\text{min} & \quad -z + 27x_2 - 12x_4 = -34 \\
\text{subject to} & \quad x_1 - x_2 + x_4 = 3 \\
& \quad x_3 - 3x_2 + 2x_4 = 2 \\
& \quad x_i \geq 0 \quad \forall i.
\end{align*}
\]

becomes

\[
\begin{align*}
\text{min} & \quad -z + 9x_2 + 6x_3 = -22 \\
\text{subject to} & \quad x_1 + 0.5x_2 - 0.5x_3 = 2 \\
& \quad x_4 - 1.5x_2 + 0.5x_3 = 1 \\
& \quad x_i \geq 0 \quad \forall i.
\end{align*}
\]

after reordering the columns.

**Note:** In a practical implementation, only quantities that are actually needed are calculated, and the quantities are calculated more efficiently. For example, the inverse matrix \( B^{-1} \) is never calculated explicitly.