1 Introduction

A network may be shared by many different services or commodities. For example:

- telephone network: each call is a commodity with its own source and sink
- transportation network: each trip is its own commodity
- supply chain: different goods move through the network

The network can be directed or undirected. We’ll assume it is a directed network $D = (V, E)$ and let $A$ denote the node-arc incidence matrix. We have $K$ commodities, each with its own sources and sinks. The vector $b^k$ gives the net demand at each node for commodity $k$. Transporting one unit of commodity $k$ on arc $(i,j) \in E$ costs $c^k_{ij}$.

Each arc $(i,j) \in E$ has a capacity $u_{ij}$. The total amount shipped through arc $(i,j) \in E$ of all the commodities must be no larger than $u_{ij}$. Without this shared bound, the problem would separate: we could solve separate min cost circulation problems for each commodity.

We’ll give two formulations: an arc-based formulation and a path-based formulation. Each formulation can be modified straightforwardly to handle the case that the network is undirected. For more details, see for example Chapter 17 of the text by Ahuja et al [1].

2 Arc-based formulation

We let $x^k_{ij}$ denote the amount of commodity $k$ shipped on arc $(i,j) \in E$. The arc-based LP formulation for the multicommodity network flow problem is:

$$
\begin{align*}
\min_{x \in \mathbb{R}^{|E| \times K}} & \sum_{k=1}^{K} \sum_{(i,j) \in E} c^k_{ij} x^k_{ij} \\
\text{subject to} & \quad Ax^k = b^k \\
& \sum_{k=1}^{K} x^k_{ij} \leq u_{ij} \quad \forall (i,j) \in E \\
& \quad x \geq 0
\end{align*}
$$

Even if all the data is integer, this problem may well have a fractional optimal solution, in contrast with the single commodity case.

Let $n = |E|$ and $m = |V|$. The arc-based formulation has $nK$ variables and $n + mK$ constraints (excluding nonnegativity constraints). For example, in a graph with 1000 nodes and 5000 arcs and with one commodity for each pair of nodes, we have on the order of $10^9$ variables and constraints. In principle, the problem can be solved using an LP solver, but in practice the size of the problem can be daunting.
### 3 A path-based formulation

We assume all the costs $c_{ij}^k$ are nonnegative, so in particular there are no negative length cycles. Any feasible flow in commodity $k$ can be represented as a sum of flows on paths. We assume each commodity has a single source $s^k$ and a single sink $t^k$; this can be done without loss of generality, by introducing a supersource and/or a supersink for each commodity and constructing capacities and costs appropriately for the edges linking the supersource to the sources and the sinks to the supersink.

We let $d^k \in \mathbb{R}$ denote the (scalar) demand for commodity $k$ and we let $P^k$ denote the set of all paths from the source for commodity $k$ to the sink for commodity $k$. Each path $p \in P^k$ has a cost per unit shipped:

$$c^k_p = \sum_{(i,j) \in p} c_{ij}^k.$$

We define an indicator parameter to capture which edges are on a path $p$:

$$\delta_{ij}(p) = \begin{cases} 
1 & \text{if } (i,j) \in p \\
0 & \text{otherwise}
\end{cases}$$

Our variables are $f^k_p$ for each $p \in P^k$, for each $k = 1, \ldots, K$, which gives the amount of commodity $k$ shipped on path $p$. The path-based formulation of the multicommodity problem is then:

$$\min_f \quad \sum_{k=1}^{K} \sum_{p \in P^k} c^k_p f^k_p$$

subject to

$$\sum_{k=1}^{K} \sum_{p \in P^k} \delta_{ij}(p) f^k_p \leq u_{ij} \quad \forall (i,j) \in E$$

$$\sum_{p \in P^k} f^k_p = d^k \quad \text{for } k = 1, \ldots, K$$

$$f^k_p \geq 0 \quad \forall p \in P^k, k = 1, \ldots, K$$

The path-based formulation has $n + K$ constraints (excluding nonnegativity constraints). For the earlier numbers of 1000 nodes and 5000 arcs and with one commodity for each pair of nodes, this results in on the order of $10^6$ constraints. The drawback is that the path-based formulation has an exponential number of variables. Thus, the paths are not all enumerated a priori; instead, they are generated as needed.

#### 3.1 The dual problem

We construct the dual to the path-based formulation LP. The dual variables are:

$$w_{ij} \quad \forall (i,j) \in E: \quad \text{dual variable to the constraint } - \left( \sum_{k=1}^{K} \sum_{p \in P^k} \delta_{ij}(p) f^k_p \right) \geq -u_{ij}$$

$$\sigma_k \quad \text{for } k = 1, \ldots, K: \quad \text{dual variable to the constraint } \sum_{p \in P^k} f^k_p = d^k$$
The dual problem is then

$$\max_{\omega, \sigma} \sum_{k=1}^{K} d_k \sigma_k - \sum_{(i,j) \in E} u_{ij} \omega_{ij}$$

subject to

$$\sigma_k - \sum_{(i,j) \in p} \omega_{ij} \leq c^k_p \quad \forall p \in P^k, k = 1, \ldots, K$$

$$w \geq 0$$

When the dual solution is chosen to satisfy complementary slackness, the dual slack is equal to the reduced cost. Let the particular dual solution be $$(\bar{\sigma}, \bar{\omega})$$. Thus, the reduced cost for primal variable $f^k_p$ is

$$\tilde{c}^k_p := c^k_p - \bar{\sigma}_k + \sum_{(i,j) \in p} \bar{\omega}_{ij} = \sum_{(i,j) \in p} (c^k_{ij} + \bar{\omega}_{ij}) - \bar{\sigma}_k$$

### 3.2 Pricing subproblem

Dual feasibility can be determined by solving a shortest path problem for each commodity $k = 1, \ldots, K$. We use modified edge costs $c^k_{ij} + \bar{\omega}_{ij}$. Intuitively, edges that are in high demand will have high shadow prices $\bar{\omega}_{ij}$, so the cost of traversing such edges is increased in the shortest path calculation. From complementary slackness, if the edge $(i, j) \in E$ is not used to capacity then $\bar{\omega}_{ij} = 0$.

If the shortest path for commodity $k$ with the modified edge weights has length less than $\bar{\sigma}_k$ then that path has a negative reduced cost.

If the shortest path for each commodity is at least $\bar{\sigma}_k$ then the primal solution $f$ and dual solution $$(\bar{\sigma}, \bar{\omega})$$ are optimal.

### 3.3 A valid lower bound

Let $v_k$ denote the length of the shortest path for commodity $k$ with arc costs $c^k_{ij} + \bar{\omega}_{ij}$. Then

$$v_k - \sum_{(i,j) \in p} \bar{\omega}_{ij} \leq c^k_p \quad \forall p \in P^k, k = 1, \ldots, K,$$

so we get a dual feasible solution by setting

$$w_{ij} = \bar{\omega}_{ij} \quad \forall (i, j) \in E, \quad \sigma_k = v_k \text{ for } k = 1, \ldots, K.$$

This leads to a lower bound on the optimal value of the multicommodity problem of

$$\text{lower bound} = \sum_{k=1}^{K} d^k v_k - \sum_{(i,j) \in E} u_{ij} \bar{\omega}_{ij}$$
3.4 Structure of basic feasible solutions

The primal constraint matrix for the path-based formulation has the structure

\[
\begin{array}{c}
\text{\(n\) constraints} \\
\hline \\
\text{\(K\) constraints} \\
\hline \\
\text{\(\ldots\)} \\
\hline
\end{array}
\]

The number of basic variables is \(n + K\). There must be at least one basic variable for each commodity, so at most \(n\) commodities use 2 or more paths. This means that we can implement the linear algebra to work only with the variables corresponding to these “extra” \(n\) paths: once these variables are calculated, the values of the remaining basic variables are determined. All calculations can be performed using a matrix of size \(n \times n\), far smaller than the full basic matrix of size \((n + K) \times (n + K)\).

This variant is known as the generalized upper bounding simplex method. For more details, see Chapter 25 of Chvátal [2].

3.5 Extensions

The approach can be extended to convex cost functions. The LPs can be solved using interior point methods.

References
