Completely Positive Programs

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Convex vs nonconvex problems

Optimization problems are often divided into convex and nonconvex problems.

Integer optimization problems are examples of nonconvex optimization problems.

The perception is that convex problems are “easy” to solve to global optimality and nonconvex problems are “hard”.

Completely positive programs are convex conic optimization problems which typically cannot be solved in polynomial time, so they are convex problems that are “hard”.

They provide exact reformulations of quadratic integer programs as convex optimization problems.
A quadratic integer program

Consider the quadratic integer program

$$\min_{x \in \mathbb{R}^n} \quad c^T x + \frac{1}{2} x^T Q x$$
subject to

$$Ax = b$$

$$x \geq 0$$

$$x_j \in \{0, 1\}, \quad j \in B \subseteq \{1, \ldots, n\}$$

(1)

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $Q$ is a symmetric $n \times n$ matrix; $Q$ need not be positive semidefinite.

$x_j \in \{0, 1\} \iff x_j(1-x_j) = 0$
Constructing a relaxation
Let \( a_i \) represent the \( i \)th row of \( A \) (written as a column vector in \( \mathbb{R}^n \)).

We follow a similar sequence of steps to those used to construct the SDP relaxation of MAXCUT.

First, we **lift**, introducing a matrix \( X \in \mathbb{R}^{n \times n} \) of variables equal to \( xx^T \):

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, X \in \mathbb{R}^{n \times n}} & \quad c^T x + \frac{1}{2} Q \cdot X \\
\text{subject to} & \quad Ax = b \\
& \quad a_i^T X a_i = b_i^2, \ i = 1, \ldots, m \\
& \quad x \geq 0 \\
& \quad x_j = X_{jj}, \ j \in B \subseteq \{1, \ldots, n\} \\
& \quad x_j \in \{0, 1\}, \ j \in B \subseteq \{1, \ldots, n\} \\
& \quad X = xx^T
\end{align*}
\]

We've exploited the linear constraint \( a_i^T x = b_i \) to construct

\[
a_i^T (x x^T) a_i = b_i^2 \iff a_i^T X a_i = b_i^2, \quad \text{since } xx^T = X.
\]
Constructing a relaxation

We can relax this problem as:

\[
\begin{aligned}
\min_{x \in \mathbb{R}^n, X \in \mathbb{R}^{n \times n}} & \quad \frac{1}{2} \begin{bmatrix} 0 & c^T \\ c & Q \end{bmatrix} \cdot \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \\
\text{subject to} & \quad Ax = b \\
& \quad a_i^T X a_i = b_i^2, \ i = 1, \ldots, m \\
& \quad x \geq 0 \\
& \quad x_j = X_{jj}, \ j \in B \subseteq \{1, \ldots, n\} \\
\end{aligned}
\]

(3)

for a convex cone \( K \).

We’ve relaxed the quadratic equality \( X = xx^T \) and introduced the final conic constraint.

If we take \( K = \mathbb{S}_+^{n+1} \), we obtain the SDP relaxation. (We have \( X \succeq xx^T \), from Schur complement.)
Schur complement:

\[ W = \begin{bmatrix} A & M \\ \Pi^T & C \end{bmatrix}, \quad A, \Pi, C \text{ symmetric.} \]

Given \( g \), minimised with \( \pi = \Pi^T \Pi^{-1} \Pi^T g \).

Form evaluates to:

\[
\begin{bmatrix} \Pi^T & g \end{bmatrix} \begin{bmatrix} A & M \\ \Pi^T & C \end{bmatrix} \begin{bmatrix} \Pi^T \\ g \end{bmatrix} = \Pi^T A \Pi + 2 \Pi^T M_2 + g^T C g = : z
\]

If \( \omega \) psd: then \( z \geq 0 \) so \( \Pi^T (C - \Pi^T A^{-1} M) \Pi \geq 0 \).

If \( \omega \) not psd: \( \exists (p, z) \) with \( z < 0 \). Setting \( p = -A^{-1} M_2 \) shows \( \Pi^T (C - \Pi^T A^{-1} M) \Pi z < 0 \).
Conic constraint:
\[
\begin{bmatrix}
1 & x^T \\
x & x^T
\end{bmatrix} \succeq 0 \iff X - x (1)^T x^T = x - xx^T \succeq 0.
\]
\[
\Rightarrow X \succeq xx^T.
\]
Completely positive matrices

We can impose stronger restrictions on $K$. For example, we can choose $K$ to be the cone of completely positive matrices.

**Definition**

The cone of $q \times q$ completely positive matrices is

$$C^P_q := \{ X \in R^{q \times q} : X = ZZ^T \text{ for some matrix } Z \text{ with every entry of } Z \text{ nonnegative} \}.$$

Note that we’d have the cone $S^q_+$ of symmetric positive semidefinite matrices if we didn’t impose the nonnegativity restriction on the entries of $Z$.

Thus, $C^P_q \subseteq S^q_+$. 

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$$d^T X d = (d^T Z)(Z^T d) = 112^T d 12^T \geq 0.$$
Examples of completely positive matrices

Since $\mathbb{CP}^q \subseteq \mathbb{S}^q_+$, taking $K = \mathbb{CP}^{n+1}$ gives a relaxation that is at least as tight as the SDP relaxation.

Two examples of matrices that are completely positive:

\[
X = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2^T \end{bmatrix}
\]

\[
X = \begin{bmatrix} 1 \\ x^T \\ xx^T \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 & x^T \end{bmatrix}
\]

provided $x \geq 0$
Properties of completely positive matrices

Some properties of $\mathcal{CP}$:

**Theorem**

1. If $X$ is completely positive then it is **doubly nonnegative**:
   - it is symmetric and positive semidefinite, and
   - every entry in $X$ is nonnegative.

2. If $X \in \mathbb{R}^{q \times q}$ is doubly nonnegative and $q \leq 4$ then $X$ is completely positive.

3. The dual cone to $\mathcal{CP}^q$ is the cone of **copositive matrices**

   $\mathcal{COP}^q := \{X \in \mathbb{R}^{q \times q} : d^T Xd \geq 0 \ \forall \ d \in \mathbb{R}_+^q\}.$

Note that $\mathbb{S}_+^q \subseteq \mathcal{COP}^q.$
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The relaxation is exact

Under some simple assumptions, the completely positive formulation (3) is equivalent to the quadratic integer program (1), so it is not just a relaxation.

Theorem (Burer, 2009)

Assume the constraints $x \geq 0$, $Ax = b$ imply $0 \leq x_j \leq 1 \ \forall j \in B$. Then the quadratic integer program (1) is equivalent to its completely positive relaxation (3), in that

1. their optimal values agree, and
2. if $(x^*, X^*)$ is optimal for (3) then $x^*$ is in the convex hull of optimal solutions to (1).
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Thus, we have a conic optimization problem that is NP-hard.

No nice barrier function is known, so it is hard to work directly with the cone of completely positive matrices.

Even the separation problem is NP-complete:

Separation problem for $\mathcal{CP}^q$: given a doubly nonnegative $X \in \mathbb{S}_+^q$, determine whether $X \in \mathcal{CP}^q$. 
Solution approaches and extensions

Cutting plane and column generation methods have been developed.

The results of Burer have been extended to show that any quadratically constrained quadratic program (convex or nonconvex) can be expressed as an equivalent completely positive program, with a slightly broadened definition of a completely positive matrix.

\[
\begin{align*}
m \in & \quad c^T x + \frac{1}{2} x^T Q x \\
\text{s.t.} & \quad A x = b \\
& \quad \sum_{i \in \mathcal{J}} x_i + \frac{1}{2} x^T Q x_i x \leq h_i \quad \forall x \\
& \quad n x \geq 1
\end{align*}
\]

Mitchell Completely Positive Programs