MATP6640/ISYE6770 Linear and Conic Optimization

Proving Strong Duality using Farkas

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We’ve previously proved the Farkas Lemma using strong duality. Here we show the converse. We consider the primal-dual pair:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax = b & (P) \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c & (D) \\
& \quad y \text{ free}
\end{align*}
\]

The Farkas Lemma states that for any matrix \( A \) and appropriately dimensioned vectors \( b \) and \( c \), exactly one of the following systems has a solution:

(I) \( Ax = b, x \geq 0 \).

(II) \( A^T y \leq 0, b^T y > 0 \).

(Note: we are going to apply the Farkas Lemma using different matrices, so the parameters in the Lemma are different from those in the primal-dual pair.)

We prove the following version of the strong duality theorem:

**Theorem 1.** If \((P)\) has a finite optimal value \( v^* \) then

1. it has an optimal solution \( x^* \), and

2. the dual problem has an optimal solution with value \( v^* \).

We have weak duality: if \( x \) is feasible in \((P)\) and \( y \) is feasible in \((D)\) then

\[
c^T x \geq y^T Ax = b^T y. \tag{1}
\]

Apply Farkas to the systems:

\[
(III) \quad \begin{bmatrix} A & 0 \\ c^T & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} b \\ v^* \end{bmatrix}, \quad \begin{bmatrix} x \\ t \end{bmatrix} \geq 0.
\]

\[
(IV) \quad \begin{bmatrix} A & 0 \\ c^T & 1 \end{bmatrix}^T \begin{bmatrix} y \\ w \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} b \\ v^* \end{bmatrix}^T \begin{bmatrix} y \\ w \end{bmatrix} > 0.
\]

Note that \((III)\) is equivalent to:

\[
(III') \quad Ax = b, c^T x \leq v^*, x \geq 0,
\]

and \((IV)\) is equivalent to

\[
(IV') \quad A^T y + wc \leq 0, w \leq 0, b^T y + wv^* > 0.
\]

If we can show system \((III')\) has a solution then that solution satisfies part (1) of the theorem. We try to prove this by contradiction and assume \((IV)\) has a solution \((\bar{y}, \bar{w})\). We can break into two cases:
1. \( \bar{w} = 0 \): Then \( A^T \bar{y} \leq 0 \) and \( b^T \bar{y} > 0 \), so for any \( x \) feasible for \((P)\) we have
\[
0 \geq (A^T \bar{y})^T x = \bar{y}^T (Ax) = b^T \bar{y} > 0,
\]
a contradiction.

2. \( \bar{w} < 0 \): Define \( \hat{y} = \frac{1}{\bar{w}} \bar{y} \). Since \((IV')\) holds, we obtain \( A^T \hat{y} \leq c \) and \( b^T y > v^* \), contradicting \((1)\).

Thus, \((IV)\) does \textbf{not} have a solution, so by Farkas system \((III)\) has a solution, so \((P)\) has an optimal solution.

We apply Farkas with a different pair of systems to show part (2) of the theorem. Consider the pair:

\[
(V) \quad \begin{bmatrix} A^T & -A^T & I & 0 \\ \bar{b}^T & -\bar{b}^T & 0 & -1 \end{bmatrix} \begin{bmatrix} y^+ \\ y^- \\ s \\ z \end{bmatrix} = \begin{bmatrix} c \\ v^* \end{bmatrix}, \begin{bmatrix} y^+ \\ y^- \\ s \\ z \end{bmatrix} \geq 0
\]

\[
(VI) \quad \begin{bmatrix} A^T & -A^T & I & 0 \\ \bar{b}^T & -\bar{b}^T & 0 & -1 \end{bmatrix}^T \begin{bmatrix} w \\ t \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} c \\ v^* \end{bmatrix}^T \begin{bmatrix} w \\ t \end{bmatrix} > 0
\]

Note that \((V)\) is equivalent to the system:

\[
(V') \quad A^T y \leq c, b^T y \geq v^*,
\]
with the change of variables \( y = y^+ - y^- \). If we can show \((V')\) holds then part (2) of the theorem is proved.

System \((VI)\) can be written as:
\[
Aw + tb \leq 0, -Aw - tb \leq 0, w \leq 0, t \geq 0, c^T w + tv^* > 0.
\]
Making the change of variables \( x = -w \), we get the equivalent system

\[
(VI') \quad Ax = tb, x \geq 0, t \geq 0, c^T x < tv^*.
\]
Assume \((VI')\) is consistent, with solution \((\bar{x}, \bar{t})\). We have two cases depending on the sign of \( \bar{t} \):

1. \( t = 0 \): Then there exists \( \bar{x} \geq 0 \) with \( A\bar{x} = 0 \) and \( c^T \bar{x} < 0 \). This gives a ray for \((P)\), so the primal problem does not have a finite optimal value: contradiction.

2. \( t > 0 \): Define \( \hat{x} = \frac{1}{t} \bar{x} \). We then have \( \hat{x} \geq 0, A\hat{x} = b \), and \( c^T \hat{x} < v^* \), contradicting the assumption that \((P)\) has finite optimal value \( v^* \).

Thus, System \((V)\) must be consistent, so there exists a dual optimal solution with value \( v^* \).