Consider the linear programming problem:

\[
\begin{align*}
\text{min } & \quad c^T x := x_1 - x_2 - 2x_3 \\
\text{s.t. } & \quad Ax := x_1 + x_2 + x_3 = 3 =: b \\
x \in X := & \begin{cases} 
0 \leq x_1 \leq 2 \\
0 \leq x_2 \\
0 \leq x_3 \leq 2
\end{cases}.
\end{align*}
\]

Thus, \(c^T = (1, -1, -2)\), \(A = (1, 1, 1)\) and \(b = 3\). The Master Problem (MP) is:

\[
\begin{align*}
\text{min } & \quad \sum_{j \in J} (c^T x^j) \lambda_j + \sum_{k \in K} (c^T d^k) \mu_k \\
\text{s.t. } & \quad \sum_{j \in J} (Ax^j) \lambda_j + \sum_{k \in K} (Ad^k) \mu_k = b \\
& \quad \sum_{j \in J} \lambda_j = 1 \\
& \quad \lambda_j \geq 0, \quad \mu_k \geq 0,
\end{align*}
\]

where \(\{x^j : j \in J\}\) is the set of extreme points of \(X\), and \(\{d^k : k \in K\}\) is the set of extreme rays of \(X\).
Initialization:

Start with the two extreme points \( x^1 := (2, 0, 0)^T \) and \( x^2 := (2, 0, 2)^T \) of \( X \).

Then when using the revised simplex method, our basic variables are \( \lambda_1 \) and \( \lambda_2 \).

So, in terms of the basic variables, we get the Revised Master Problem (RMP):

\[
\begin{align*}
\min & \quad 2\lambda_1 - 2\lambda_2 \\
\text{s.t.} & \quad 2\lambda_1 + 4\lambda_2 = 3 \\
& \quad \lambda_1 + \lambda_2 = 1 \\
& \quad \lambda_j \geq 0.
\end{align*}
\]

Here, the coefficient of \( \lambda_1 \) in the objective function is \( c^T x^1 \), the coefficient of 4 for \( \lambda_2 \) in the first constraint comes from \( Ax^2 \), etc. The second constraint is the convexity constraint \( \sum_{j \in J} \lambda_j = 1 \). We will denote costs in (MP) by \( \hat{c}_j \), columns by \( \hat{a}_j \), etc.

Initial basis:

\[
\begin{pmatrix}
\lambda_1 \\
\lambda_2
\end{pmatrix}
= 
\begin{pmatrix}
2 & 4 \\
1 & 1
\end{pmatrix}
, \quad
\hat{B}^{-1} = 
\begin{pmatrix}
-0.5 & 2 \\
0.5 & -1
\end{pmatrix}
.
\]

Thus, the initial basic feasible solution is:

\[
\begin{pmatrix}
\bar{\lambda}_1 \\
\bar{\lambda}_2
\end{pmatrix}
= 
\hat{B}^{-1} \hat{b}
= 
\begin{pmatrix}
0.5 \\
0.5
\end{pmatrix}
.
\]

the corresponding dual solution is:

\[
(\pi, \sigma) = \hat{c}_B^T \hat{B}^{-1} = (-2, 6);
\]

the corresponding solution to the initial problem is

\[
\bar{x} = \bar{\lambda}_1 x^1 + \bar{\lambda}_2 x^2 = (2, 0, 1)^T.
\]

Thus the subproblem is

\[
\begin{align*}
\min & \quad z := (c^T - \pi^T A)x = ((1, -1, -2) + 2(1, 1, 1))x = 3x_1 + x_2 \\
\text{s.t.} & \quad x \in X.
\end{align*}
\]

By inspection, the optimal solution is \( x = (0, 0, 0)^T \), giving \( z^* = 0 < 6 = \sigma \).

Therefore, our current primal solution is not optimal, and we need to introduce a column for \( x_3 := (0, 0, 0)^T \) into the Revised Master Problem.
• First iteration:

Then $\lambda_3$ enters the basis. Objective function value is $\hat{c}_3 = c^T x^3 = 0$; column of constraint matrix is $\hat{a}^3 = \begin{pmatrix} A x^3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ In order to determine which variable leaves the basis, we need to calculate $\hat{B}^{-1} \hat{a}^3 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. We then compare this column of “$B^{-1} N$” with the current bfs $\hat{x}_B = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ using the minimum ratio test. Thus, $\lambda_1$ leaves the basis. The new basis matrix is:

$$
\begin{align*}
\lambda_3 : & \quad \hat{B} = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} \\
\lambda_2 : & \quad \hat{B}^{-1} = \begin{pmatrix} -0.25 & 1 \\ 0.25 & 0 \end{pmatrix}
\end{align*}
$$

Thus, the new primal solution to $(MP)$ is

$$(\bar{x}, \bar{x}) = \hat{c}_B^T \hat{B}^{-1} = (0, -2) \hat{B}^{-1}(3, 1)^T = (0.25, 0.75)^T$$

and the new dual solution is

$$(\pi, \sigma) = \hat{c}_B^T \hat{B}^{-1} = (0, -2) \hat{B}^{-1} = (-0.5, 0).$$

Hence, the new solution for the original LP is:

$$
\bar{x} = \bar{x}_3 x^3 + \bar{x}_2 x^2 = (1.5, 0, 1.5)^T.
$$

The subproblem is

$$
\min \quad z = (c^T - \pi A)x = ((1, -1, -2) + 0.5(1, 1, 1))x = (1.5, -0.5, -1.5)x \\
\text{s.t.} \quad x \in X.
$$

This subproblem has an unbounded optimal solution, and the extreme ray is $d = (0, 1, 0)^T$.

Set $d^1 := (0, 1, 0)^T$; then $\hat{c}_4 = c^T d^4 = -1$ and $\hat{a}_4 = \begin{pmatrix} A d^4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We introduce $\mu_4$ into the basis.
• **Second iteration:**

We need to determine which variable leaves the basis. Therefore, we calculate \( \hat{B}^{-1} \hat{a}_4 = (-0.25, 0.25)^T \). Using the minimum ratio test with \( \hat{x}_B = (0.25, 0.75)^T \) shows that \( \lambda_2 \) leaves the basis. The new basis matrix is:

\[
\hat{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \hat{B}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

Thus, the new primal solution to \((MP)\) is

\[
(\tilde{\lambda}_3, \tilde{\mu}_4)^T = \hat{B}^{-1} \hat{\bar{b}} = \hat{B}^{-1} (3, 1)^T = (1, 3)^T
\]

and the new dual solution is

\[
(\pi, \sigma) = \hat{c}_B^T \hat{B}^{-1} = (0, -1) \hat{B}^{-1} = (-1, 0).
\]

Hence, the new solution for the original LP is:

\[
\bar{x} = \tilde{\lambda}_3 x^3 + \tilde{\mu}_4 d^4 = (0, 3, 0)^T.
\]

The subproblem is

\[
\text{min } z = (c^T - \pi A)x = ((1, -1, -2) + 1(1, 1, 1))x = 2x_1 - x_3 \\
\text{s.t. } x \in X.
\]

An optimal solution is \( x = (0, 0, 2)^T \), giving \( z^* = -2 < \sigma \). So we do not have the optimal solution to the *master problem* \((MP)\). Hence, we set \( x^5 := (0, 0, 2)^T \), and we introduce a column for this extreme point into the *Revised Master Problem*. 


• **Third iteration:**

Then $\lambda_5$ enters the basis. Objective function value is $\hat{c}_5 = c^T x^5 = -4$; column of constraint matrix is $\hat{a}^5 = \begin{pmatrix} Ax^5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. In order to determine which variable leaves the basis, we need to calculate $\hat{B}^{-1} \hat{a}^5 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Using the minimum ratio test with $\hat{x}_B = (1, 3)^T$ shows that $\lambda_3$ leaves the basis. The new basis matrix is:

$$
\begin{align*}
\lambda_5 : \\
\mu_4 : \\
\hat{B} &= \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}; \\
\hat{B}^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.
\end{align*}
$$

Thus, the new primal solution to $(MP)$ is

$$(\bar{\lambda}_5, \bar{\mu}_4)^T = \hat{B}^{-1} \hat{b} = \hat{B}^{-1} (3, 1)^T = (1, 1)^T$$

and the new dual solution is

$$(\pi, \sigma) = \hat{c}_B^T \hat{B}^{-1} = (-4, -1) \hat{B}^{-1} = (-1, -2).$$

Hence, the new solution for the original LP is:

$$\bar{x} = \bar{\lambda}_5 x^5 + \bar{\mu}_4 d^4 = (0, 1, 2)^T.$$  

The subproblem is

$$
\begin{align*}
\min \quad & z = (c^T - \pi A)x = ((1, -1, -2) + 1(1, 1, 1))x = 2x_1 - x_3 \\
\text{s.t.} \quad & x \in X.
\end{align*}
$$

An optimal solution is $x = (0, 0, 2)^T$, giving $z^* = -2 = \sigma$. Hence, our optimality criterion is satisfied, and $\bar{x} = (0, 1, 2)^T$ is an optimal solution to our original LP.
Dantzig-Wolfe example