Consider a linear programming problem:

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \in X := \{x \in \mathbb{R}^n : Hx = h, x \geq 0\} \neq \emptyset.
\end{align*}
\]  

The variables \(x \in \mathbb{R}^n\). The matrix \(A\) is \(m_A \times n\), the matrix \(H\) is \(m_H \times n\), and all vectors are dimensioned appropriately. Let the extreme points and extreme rays of \(X\) be

- extreme points: \(\{x^j : j \in J\}\),
- extreme rays: \(\{d^k : k \in K\}\).

**Definition 1.** A ray \(d\) of a pointed polyhedron \(X = \{x \in \mathbb{R}^n : Hx = h, x \geq 0\}\) satisfies \(Hd = 0\), \(d \geq 0\), and \(d \neq 0\). It is an extreme ray if any pair of nonzero vectors \(g, h \in X\) satisfying \(g + h = d\) must also each be scalar multiples of \(d\).

Every point in \(X\) has the form

\[
x = \sum_{j \in J} \lambda_j x^j + \sum_{k \in K} \mu_k d^k
\]  

for nonnegative multipliers \(\lambda_j : j \in J\) and \(\mu_k : k \in K\) satisfying

\[
\sum_{j \in J} \lambda_j = 1.
\]

Instead of optimizing directly over \(x\), we can optimize over the multipliers \(\lambda\) and \(\mu\). Replacing \(x\) in \((P)\) by a sum over the extreme points and extreme rays leads to the Master Problem (MP):

\[
\begin{align*}
\min_{\lambda, \mu} & \quad \sum_{j \in J} (c^T x^j) \lambda_j + \sum_{k \in K} (c^T d^k) \mu_k \\
\text{s.t.} & \quad \sum_{j \in J} (Ax^j) \lambda_j + \sum_{k \in K} (Ad^k) \mu_k = b \\
& \quad \lambda_j \geq 0 \forall j \in J, \quad \mu_k \geq 0 \forall k \in K.
\end{align*}
\]  

This problem has fewer constraints than \((P)\), but typically far more variables. We introduce dual variables \(\pi \in \mathbb{R}^{m_A}\) corresponding to the first set of equality constraints in \((MP)\), and a dual variable \(\sigma\) corresponding to the last equality constraint in \((MP)\). The dual problem is then

\[
\begin{align*}
\max_{\pi, \sigma} & \quad b^T \pi + \sigma \\
\text{subject to} & \quad (Ax^j)^T \pi + \sigma \leq c^T x^j, \quad j \in J \\
& \quad (Ad^k)^T \pi \leq c^T d^k, \quad k \in K \\
& \quad \pi, \sigma \text{ free}
\end{align*}
\]  

Because there is a large number of extreme points and rays of \(X\), they are typically generated as needed. We work with subsets of \(J\) and \(K\). Optimal primal and dual solutions are found for the current subset, giving the primal solution \((\bar{\lambda}, \bar{\mu})\) and dual solution \((\bar{\pi}, \bar{\sigma})\). The primal solution is feasible in the full \((MP)\): give all the extra variables the value 0. We need to check dual feasibility. Instead of checking each extreme point and extreme ray separately, we can solve an LP subproblem to determine dual feasibility.

We exploit the following theorem:
Theorem 1. The solution $(\bar{\pi}, \bar{\sigma})$ is feasible in $(MD)$ if and only if $(c - A^T \bar{\pi})^T x \geq \sigma$ for all $x \in X$.

Proof. First, assume $(\bar{\pi}, \bar{\sigma})$ is feasible in $(MD)$:
Note from (1) that any $x \in X$ satisfies
\begin{align*}
(c - A^T \bar{\pi})^T x &= (c - A^T \bar{\pi})^T \left( \sum_{j \in J} \lambda_j x^j + \sum_{k \in K} \mu_k d^k \right) \\
&= \sum_{j \in J} \lambda_j (c - A^T \bar{\pi})^T x^j + \sum_{k \in K} \mu_k (c - A^T \bar{\pi})^T d^k \\
&\geq \sum_{j \in J} \lambda_j \bar{\sigma} + \sum_{k \in K} 0 \quad \text{since $(\bar{\pi}, \bar{\sigma})$ is feasible in $(MD)$} \\
&= \bar{\sigma} \quad \text{since } \sum_{j \in J} \lambda_j = 1,
\end{align*}
as required.

Second, assume $(\bar{\pi}, \bar{\sigma})$ is not feasible in $(MD)$:
If a constraint $(Ax^j)^T \pi + \sigma \leq c^T x^j$ is violated for some $j \in J$ by $(\bar{\pi}, \bar{\sigma})$ then we have
\[(c - A^T \bar{\pi})^T x^j < \bar{\sigma}\]
for the point $x^j \in X$, so the required inequality does not hold for some $x \in X$. If a constraint $(Ad^k)^T \pi \leq c^T d^k$ is violated for some $k \in K$ by $(\bar{\pi}, \bar{\sigma})$ then we have
\[(c - A^T \bar{\pi})^T d^k < 0.\]
Let $\tilde{x} \in X$. For sufficiently large $\alpha > 0$ we have
\[(c - A^T \bar{\pi})^T (\tilde{x} + \alpha d^k) < \bar{\sigma}\]
for the point $\tilde{x} + \alpha d^k \in X$.

Thus, we can determine feasibility in $(MD)$ by setting up a subproblem:
\[
\begin{align*}
\min_{x} & \quad v := (c - A^T \bar{\pi})^T x \\
\text{subject to} & \quad x \in X \\
& \quad (SP(\bar{\pi}))
\end{align*}
\]

There are three possible outcomes to the subproblem:

1. Optimal value $v$ of $(SP(\bar{\pi}))$ is $v < \bar{\sigma}$: Then we have found another extreme point of $X$. The dual solution $(\bar{\pi}, \bar{\sigma})$ violates the constraint in $(MD)$ corresponding to this extreme point; the corresponding $\lambda_j$ in $(MP)$ has a negative reduced cost.

2. Optimal value $v$ of $(SP(\bar{\pi}))$ is $v \geq \bar{\sigma}$: The dual solution $(\bar{\pi}, \bar{\sigma})$ is feasible in $(MD)$, so we have solved $(MP)$ and $(MD)$, and hence $(P)$.

3. Optimal value of $(SP(\bar{\pi}))$ is unbounded: Then we have found another ray $d$ of $X$. The dual solution $(\bar{\pi}, \bar{\sigma})$ violates the constraint in $(MD)$ corresponding to this extreme ray; the corresponding $\mu_k$ in $(MP)$ has a negative reduced cost.

In Cases 1 and 3, we update the current subset of columns used in the Master Problem and iterate; in Case 2 we terminate. Since $X$ has a finite number of extreme points and rays, the algorithm terminates in a finite number of iterations.