Optimization problems are often divided into convex and nonconvex problems. Integer optimization problems are examples of nonconvex optimization problems. The perception is that convex problems are “easy” to solve to global optimality and nonconvex problems are “hard”. Completely positive programs are convex conic optimization problems which typically cannot be solved in polynomial time, so they are convex problems that are “hard”. They provide exact reformulations of quadratic integer programs as convex optimization problems.

Consider the quadratic integer program
\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x + \frac{1}{2} x^T Q x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0 \\
& \quad x_j \in \{0, 1\}, j \in B \subseteq \{1, \ldots, n\}
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, \) and \( Q \) is a symmetric \( n \times n \) matrix; \( Q \) need not be positive semidefinite. Let \( a_i \) represent the \( i \)th row of \( A \) (written as a column vector in \( \mathbb{R}^n \)). We follow a similar sequence of steps to those used to construct the SDP relaxation of \( \text{MaxCut} \). First, we lift, introducing a matrix \( X \in \mathbb{R}^{n \times n} \) of variables equal to \( xx^T \):

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, X \in \mathbb{R}^{n \times n}} & \quad c^T x + \frac{1}{2} x^T Q x \\
\text{subject to} & \quad Ax = b \\
& \quad a_i^T X a_i = b_i^2, i = 1, \ldots, m \\
& \quad x \geq 0 \\
& \quad x_j = X_{jj}, j \in B \subseteq \{1, \ldots, n\} \\
& \quad x_j \in \{0, 1\}, j \in B \subseteq \{1, \ldots, n\} \\
& \quad X = xx^T
\end{align*}
\]

We can relax this problem as:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, X \in \mathbb{R}^{n \times n}} & \quad \frac{1}{2} \begin{bmatrix} 0 & c^T \\
\end{bmatrix} \cdot \begin{bmatrix} 1 & x^T \\
x & X \\
\end{bmatrix} \\
\text{subject to} & \quad Ax = b \\
& \quad a_i^T X a_i = b_i^2, i = 1, \ldots, m \\
& \quad x \geq 0 \\
& \quad x_j = X_{jj}, j \in B \subseteq \{1, \ldots, n\} \\
& \quad \begin{bmatrix} 1 & x^T \\
x & X \\
\end{bmatrix} \in K
\end{align*}
\]

for a convex cone \( K \). If we take \( K = S_+^{n+1} \), we obtain the SDP relaxation. We can impose stronger restrictions on \( K \). For example, we can choose \( K \) to be the cone of completely positive matrices.
Definition 1. The cone of $q \times q$ completely positive matrices is

$$\mathcal{CP}^q := \{ X \in \mathbb{R}^{q \times q} : X = ZZ^T \text{ for some matrix } Z \text{ with every entry of } Z \text{ nonnegative} \}.$$ 

It is easy to see that $\mathcal{CP}^n \subseteq S^n_+$, so taking $K = \mathcal{CP}^{n+1}$ gives a relaxation that is at least as tight as the SDP relaxation. Two examples of matrices that are completely positive:

$$X = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ x^T \\ xx^T \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 & x^T \end{bmatrix}$$

Some properties of $\mathcal{CP}$:

Theorem 1. 1. If $X$ is completely positive then it is doubly nonnegative:

(a) it is symmetric and positive semidefinite, and

(b) every entry in $X$ is nonnegative.

2. If $X \in \mathbb{R}^{q \times q}$ is doubly nonnegative and $q \leq 4$ then $X$ is completely positive.

3. The dual cone to $\mathcal{CP}^q$ is the cone of copositive matrices

$$\mathcal{COP}^q := \{ X \in \mathbb{R}^{q \times q} : d^T X d \geq 0 \ \forall \ d \in \mathbb{R}_+^q \}.$$ 

Under some simple assumptions, (3) is equivalent to (1), so it is not just a relaxation.

Theorem 2 (Burer, 2009). Assume the constraints $x \geq 0$, $Ax = b$ imply $0 \leq x_j \leq 1 \ \forall \ j \in B$. Then the quadratic integer program [1] is equivalent to its completely positive relaxation [3], in that

1. their optimal values agree, and

2. if $(x^*, X^*)$ is optimal for [3] then $x^*$ is in the convex hull of optimal solutions to [1].

Thus, we have a conic optimization problem that is NP-hard. No nice barrier function is known, so it is hard to work directly with the cone of completely positive matrices. Even the separation problem is NP-complete:

Separation problem for $\mathcal{CP}^q$: given a doubly nonnegative $X \in \mathcal{CP}^q$, determine whether $X \in \mathcal{CP}^q$.

Cutting plane and column generation methods have been developed.

The results of Burer have been extended to show that any quadratically constrained quadratic program (convex or nonconvex) can be expressed as an equivalent completely positive program, with a slightly broadened definition of a completely positive matrix.