We write our standard form semidefinite program as

$$\min_X \quad C \cdot X$$

subject to

$$A_i \cdot X = b_i, \quad i = 1, \ldots, m$$

$$X \succeq 0$$  

(1)

where:

- $C$ is an $n \times n$ symmetric matrix
- $A_i$ is an $n \times n$ symmetric matrix for $i = 1, \ldots, m$
- $b_i$ is a scalar for $i = 1, \ldots, m$.

The parameter matrices $C$ and $A_i$ need not be positive semidefinite, although they are assumed to be symmetric. Recall that $C \cdot X$ represents the Frobenius inner product between the symmetric matrices $C$ and $X$, which is equal to the trace($CX$). The dual problem is

$$\max_{y \in \mathbb{R}^m, S \in \mathbb{S}^n_+} \quad b^T y$$

subject to

$$\sum_{i=1}^m y_i A_i + S = C$$

$$S \succeq 0$$  

(2)

Notice that the dual slack variables $S = C - \sum_{i=1}^m y_i A_i$ constitute a symmetric positive semidefinite matrix in feasible dual solutions.

**Example 1.** We look at the SDP relaxation of the MAXCUT problem on the following graph:

```
  2
 / \   \
|   |   |
1-----3
 |
 |
1
```

The SDP relaxation is written

$$\max_X \quad 0.25 \text{trace}(L_G X)$$

subject to

$$X_{ii} = 1, \quad \text{for } i = 1, \ldots, n$$

$$X \succeq 0, \quad \text{for } i = 1, \ldots, n$$  

(3)
where $L_G$ is the Laplacian matrix. In our case, we have

$$L_G = \begin{bmatrix} 9 & -6 & -3 \\ -6 & 7 & -1 \\ -3 & -1 & 4 \end{bmatrix}$$

Turning the problem into an equivalent minimization problem and rescaling, the SDP relaxation of the MaxCut instance is equivalent to the problem

$$\begin{align*}
\min_{X \in S_3^+} & \quad \begin{bmatrix} -9 & 6 & 3 \\ 6 & -7 & 1 \\ 3 & 1 & -4 \end{bmatrix} \cdot X \\
\text{subject to} & \quad X_{11} = 1 \\
& \quad X_{22} = 1 \\
& \quad X_{33} = 1 \\
& \quad X \succeq 0
\end{align*}$$

so the dual is

$$\begin{align*}
\max_{y \in \mathbb{R}^3, S \in S_3^+} & \quad y_1 + y_2 + y_3 \\
\text{subject to} & \quad \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} + S = \begin{bmatrix} -9 & 6 & 3 \\ 6 & -7 & 1 \\ 3 & 1 & -4 \end{bmatrix} \\
& \quad S \succeq 0
\end{align*}$$

The optimal solution to the MaxCut instance is to set $V_1 = \{1\}$, $V_2 = \{2, 3\}$, corresponding to $\bar{x} = (1, -1, -1)^T$ and then

$$\bar{X} = \bar{x}\bar{x}^T = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

with value $-36$. One feasible dual solution is $y = (-17, -13, -7)$, with value $-37$. It's feasible, since the resulting dual slack matrix is

$$S = \begin{bmatrix} -9 & 6 & 3 \\ 6 & -7 & 1 \\ 3 & 1 & -4 \end{bmatrix} - \begin{bmatrix} -17 & 0 & 0 \\ 0 & -13 & 0 \\ 0 & 0 & -7 \end{bmatrix} = \begin{bmatrix} 8 & 6 & 3 \\ 6 & 6 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

which has determinant 10, and all principal subdeterminants nonnegative.

Exercise: Find the optimal solution to this SDP.
Example 2. An example with a duality gap:

\[
\begin{align*}
\min_{X \in S_+^3} & \quad X_{33} \\
\text{subject to} & \quad X_{22} = 0 \\
& \quad X_{12} + X_{21} + X_{33} = 1 \\
& \quad X \succeq 0
\end{align*}
\]

Note that the first equality constraint then forces \(X_{12} = X_{21} = 0\) in order for \(X \succeq 0\) to hold. Then every feasible solution must have \(X_{33} = 1\), so the optimal primal value is 1.

The dual problem is

\[
\begin{align*}
\max_{y \in \mathbb{R}^2, S \in S_+^3} & \quad y_2 \\
\text{subject to} & \quad y_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
& \quad S \succeq 0
\end{align*}
\]

so

\[
S = \begin{bmatrix} 0 & -y_2 & 0 \\ -y_2 & -y_1 & 0 \\ 0 & 0 & 1 - y_2 \end{bmatrix}
\]

The problem is feasible, for example we can take \(y = (0, 0)\). Since \(S_{11} = 0\), we must have \(y_2 = 0\), so the optimal dual value is 0.

Example 3. An example which is primal infeasible and has a finite dual optimal value:

\[
\begin{align*}
\min_{X \in S_+^2} & \quad 0 \\
\text{subject to} & \quad X_{11} = 0 \\
& \quad X_{12} + X_{21} = 2 \\
& \quad X \succeq 0
\end{align*}
\]

Since \(X_{11} = 0\), the requirement \(X \succeq 0\) forces \(X_{12} = X_{21} = 0\), but then the second linear constraint is violated.

The dual problem is

\[
\begin{align*}
\max_{y \in \mathbb{R}^2, S \in S_+^2} & \quad 2y_2 \\
\text{subject to} & \quad y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
& \quad S \succeq 0
\end{align*}
\]

so

\[
S = \begin{bmatrix} -y_1 & -y_2 \\ -y_2 & 0 \end{bmatrix}
\]

Any feasible solution must have \(y_2 = 0\) so the dual has a finite optimal value of 0.
Example 4. Example 3 is close to a feasible primal problem:
\[
\begin{align*}
\min_{X \in S_+^2} & \quad 0 \\
\text{subject to} & \quad X_{11} = \epsilon \\
& \quad X_{12} + X_{21} = 2 \\
& \quad X \succeq 0
\end{align*}
\]
where \( \epsilon \) is a positive parameter. This has feasible solutions including 
\[
X = \begin{bmatrix}
\epsilon & 1 \\
1 & 1/\epsilon
\end{bmatrix}
\]

The dual problem is
\[
\begin{align*}
\max_{y \in \mathbb{R}^2, S \in S_+^2} & \quad \epsilon y_1 + 2y_2 \\
\text{subject to} & \quad y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
& \quad S \succeq 0
\end{align*}
\]
so
\[
S = \begin{bmatrix}
-y_1 & -y_2 \\
-y_2 & 0
\end{bmatrix}
\]
Any feasible solution must have \( y_2 = 0 \) and \( y_1 \leq 0 \) so the dual has a finite optimal value of 0.

Example 5. An example with an irrational optimal value:
\[
\begin{align*}
\min_{X \in S_+^2} & \quad X_{12} + X_{21} \\
\text{subject to} & \quad X_{11} = 1 \\
& \quad X_{22} = 2 \\
& \quad X \succeq 0
\end{align*}
\]
The optimal value is \(-2\sqrt{2}\) and the optimal solution is
\[
X = \begin{bmatrix}
1 & -\sqrt{2} \\
-\sqrt{2} & 2
\end{bmatrix}
\]
The dual problem is
\[
\begin{align*}
\max_{y \in \mathbb{R}^2, S \in S_+^2} & \quad y_1 + 2y_2 \\
\text{subject to} & \quad y_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
& \quad S \succeq 0
\end{align*}
\]
so
\[
S = \begin{bmatrix}
-y_1 & 1 \\
1 & -y_2
\end{bmatrix}
\]
The optimal dual solution is \( y = (-\sqrt{2}, -1/\sqrt{2}) \) with optimal value of \(-2\sqrt{2}\).