We have a network with arc capacities. We want to increase the capacities in order to meet future demand. Future demand is uncertain. What is the most cost-effective way to increase the capacity of the network?

**Current demand and capacity**

![Diagram of network with nodes r, t, u, v and capacities and demand](image)

We assume there are four scenarios for future demand:

<table>
<thead>
<tr>
<th>scenario $s$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability $p_s$</td>
<td>0.3</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
</tr>
<tr>
<td>demand $d_u^s$ at $u$</td>
<td>15</td>
<td>30</td>
<td>25</td>
<td>20</td>
</tr>
<tr>
<td>demand $d_v^s$ at $v$</td>
<td>8</td>
<td>20</td>
<td>10</td>
<td>15</td>
</tr>
</tbody>
</table>

The expansion costs are as follows:

<table>
<thead>
<tr>
<th>edge</th>
<th>cost per unit increase</th>
<th>maximum possible increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(r, u)$</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>$(r, t)$</td>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>$(r, v)$</td>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

The cost per unit of unmet demand is 8. The objective is to minimize the sum of the cost of expansion and the expected cost of unmet demand. The expansion decisions have to be made before demand is known.

First stage decision variables:

$$x_{ij} = \text{increase in capacity of arc } (i, j)$$
Master Problem at iteration $K$ (aggregated):

$$\min_{x,t} \quad 4x_{ru} + 5x_{rt} + 3x_{rv} + t$$

subject to  \quad 0 \leq x_{ij} \leq 12 \text{ for edges } (r,u), (r,v), (r,t) \quad (\zeta^k)^T (x - x^k) + t \geq Q(x^k) \text{ for } k = 1, 2, \ldots, K - 1

Second stage decision variables:

\begin{align*}
y_{ij}^s & = \text{ flow on arc } (i,j) \text{ in scenario } s \\
z_j^s & = \text{ unmet demand at node } j \text{ in scenario } s
\end{align*}

The subproblems can be written

$$Q(x, \xi^s) = \min_{y,z} \quad 8(z_u^s + z_v^s)$$

subject to

\begin{align*}
y_{ru}^s & \leq 5 + x_{ru} \\
y_{rv}^s & \leq 7 + x_{rv} \\
y_{rt}^s & \leq 11 + x_{rt} \\
y_{tu}^s & \leq 12 \\
y_{tv}^s & \leq 16 \\
y_{tu}^s + y_{tv}^s & = y_{rt}^s \\
y_{ru}^s + y_{ta}^s & = d_u^s - z_u^s \\
y_{rv}^s + y_{tv}^s & = d_v^s - z_v^s \\
y_{ij}^s & \geq 0 \quad \text{for each edge } (i,j), \\
z_i^s & \geq 0 \quad \text{for } i = u, i = v,
\end{align*}

Each subproblem has 7 variables and 8 constraints, plus simple bound constraints. Let $\pi$ be the dual variables corresponding to the inequality constraints, and $\sigma$ the dual variables corresponding to the equality constraints. The dual of the subproblem is:

$$Q(x, \xi^s) = \max_{\pi, \sigma} \quad -\pi_{ru}^s (5 + x_{ru}) - \pi_{rv}^s (7 + x_{rv}) - \pi_{rt}^s (11 + x_{rt}) - 12\pi_{tu}^s - 16\pi_{tv}^s + d_u^s \sigma_u^s + d_v^s \sigma_v^s$$

subject to

\begin{align*}
\sigma_u^s - \pi_{ru}^s & \leq 0 \\
\sigma_v^s - \pi_{rv}^s & \leq 0 \\
\sigma_t^s - \pi_{rt}^s & \leq 0 \\
-\sigma_u^s + \sigma_u^s - \pi_{tu}^s & \leq 0 \\
-\sigma_t^s + \sigma_v^s - \pi_{tv}^s & \leq 0 \\
\sigma_u^s & \leq 8 \\
\sigma_v^s & \leq 8 \\
\pi_{ij}^s & \geq 0 \quad \text{for each edge } (i,j), \\
\sigma_i^s & \text{ free } \text{ for } i = t, u, v
\end{align*}

Eg, the optimal solution in $s = 2$ with $x = 0$ has $Q(x, \xi^2) = 216$ and:

$$\pi_{ru}^s = \pi_{rv}^s = \pi_{rt}^s = \sigma_u^s = \sigma_v^s = \sigma_t^s = 8, \pi_{tu}^s = \pi_{tv}^s = 0.$$ 

The resulting constraint in the **disaggregated** form would be:

$$8x_{ru} + 8x_{rv} + 8x_{rt} + t_s \geq 216.$$