The Weyl-Minkowski Theorems

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February 2018
Introduction

There are two fundamentally different ways to look at polyhedra:

- **As constrained polyhedra:** \( P = \{ x \in \mathbb{R}^n : Ax \geq b \} \).

- **As finitely generated polyhedra:**
  \( Q = \{ x \in \mathbb{R}^n : x = Dy, \text{ with possible additional linear constraints on } y \} \).
  The columns of \( D \) are the *generators* of \( Q \).

The Weyl and Minkowski Theorems show that any polyhedron that is represented in one form can also be represented in the other form.

The theorems are first stated and proved for the case of convex cones and then the results are extended out to more general polyhedra.
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Example

\[ \left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} 4 & 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 2 & 8 \end{bmatrix} y, \ y \in \mathbb{R}^5, \ y \geq 0, \ y_3 + y_4 + y_5 = 1 \right\} \]

\[ = \left\{ x \in \mathbb{R}^2 : x_1 + 2x_2 \geq 7, \ 3x_1 + x_2 \geq 11, \ -x_1 + 4x_2 \geq -1, \ 2x_1 - 2x_2 \geq -14 \right\} \]
Cones

Constrained and finitely generated convex cones have the forms:

- Constrained: $K = \{ x \in \mathbb{R}^n : Ax \geq 0 \}$ for some matrix $A$.
- Finitely generated: $K = \{ x \in \mathbb{R}^n : x = Dy \text{ for some } y \geq 0 \}$. 

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Duals of sets

Recall that the dual of a set $M \subseteq \mathbb{R}^n$ is

$$M^+ := \{ y \in \mathbb{R}^n : y^T x \geq 0 \ \forall \ x \in M \}.$$  

The next proposition is left as an exercise:

**Proposition**

Let $S \subseteq \mathbb{R}^n$. We have $S = S^{++}$ if and only if $S$ is a constrained cone.
Examples of dual cones

Find dual cones to:

- $\mathbb{R}^n$.
- $\mathbb{R}_+^n$.
- Cone of symmetric positive semidefinite matrices.
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Theorem (Weyl)

Any finitely generated convex cone is finitely constrained.

Proof.

Let \( K \) be a finitely generated cone. Then for some matrix \( D \) we have

\[
K = \{ x \in \mathbb{R}^n : x = Dy \text{ for some } y \geq 0 \}
\]
\[
= \{ x \in \mathbb{R}^n : \text{the system } x - Dy = 0, y \geq 0 \text{ is consistent in } x, y \}
\]
\[
= \{ x \in \mathbb{R}^n : Bx \geq 0 \} \quad \text{for some matrix } B,
\]

by using Fourier-Motzkin elimination to eliminate \( y \).
**Theorem (Weyl)**

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□
Corollary

Given $A \in \mathbb{R}^{m \times n}$, let $K := \{Ax : x \geq 0\} \subseteq \mathbb{R}^m$ and $L := \{y \in \mathbb{R}^m : A^Ty \geq 0\}$. Then $K^+ = L$ and $L^+ = K$.

Proof.

We first show $K^+ = L$. We have

\[
K^+ = \{z \in \mathbb{R}^m : z^Tw \geq 0 \forall w \in K\}
\]
\[
= \{z \in \mathbb{R}^m : z^TAx \geq 0 \forall x \geq 0\}
\]
\[
= \{z \in \mathbb{R}^m : (A^Tz)^Tx \geq 0 \forall x \geq 0\}
\]

which holds if and only if $A^Tz \geq 0$, so $K^+ = L$. \qed
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Proof.

We first show $K^+ = L$. We have

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K^+ = \{z \in \mathbb{R}^m : z^T w \geq 0 \ \forall w \in K\}
= \{z \in \mathbb{R}^m : z^T Ax \geq 0 \ \forall x \geq 0\}
= \{z \in \mathbb{R}^m : (A^T z)^T x \geq 0 \ \forall x \geq 0\}
\]

which holds if and only if $A^T z \geq 0$, so $K^+ = L$. 

\[\square\]
Proof of second half

**Corollary**

Given \( A \in \mathbb{R}^{m \times n} \), let \( K := \{ Ax : x \geq 0 \} \subseteq \mathbb{R}^m \) and \( L := \{ y \in \mathbb{R}^m : A^T y \geq 0 \} \). Then \( K^+ = L \) and \( L^+ = K \).

**Proof.**

Now we show \( L^+ = K \). By Weyl’s Theorem, the cone \( K \) can also be expressed as a constrained cone, so there is a matrix \( C \) with \( K = \{ x \in \mathbb{R}^n : Cx \geq 0 \} \). From Proposition 1, we have \( K = K^{++} = (K^+)^+ = L^+ \), as required.
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Corollary

Given $A \in \mathbb{R}^{m \times n}$, let $K := \{Ax : x \geq 0\} \subseteq \mathbb{R}^m$ and $L := \{y \in \mathbb{R}^m : A^T y \geq 0\}$. Then $K^+ = L$ and $L^+ = K$.

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Corollary (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following systems has a solution:

(I) $Ax = b, \ x \geq 0$.

(II) $A^T y \leq 0, \ b^T y > 0$.

Proof.

Define $K$ and $L$ as in Corollary 2. First, assume (I) fails. Then $b \not\in K$. By Weyl’s Theorem and Proposition 1, we have $b \not\in K^{++} = (K^+)^+$. So there exists $y \in K^+$ with $b^T y < 0$. From Corollary 2, we have $K^+ = \{y \in \mathbb{R}^m : A^T y \geq 0\}$, so (II) is consistent.

Now, assume (II) holds, so $\exists y \in L$ with $b^T y < 0$. So $b \not\in L^+ = K$ by Corollary 2, so (I) fails.
Farkas

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Now, assume (II) holds, so $\exists y \in L$ with $b^T y < 0$. So $b \notin L^+ = K$ by Corollary 2, so (I) fails.
Minkowski for Cones

**Corollary (Minkowski’s Theorem)**

*Any finitely constrained cone is nonempty and finitely generated.*

**Proof.**

Define $K$ and $L$ as in Corollary 2. We need to show that the generic finitely constrained cone $L$ is finitely constrained and nonempty. Note first that the origin is a point in $L$, so $L$ is nonempty. Now we need to show that $L$ is finitely generated. We know from Corollary 2 that $L^+ = K$, a finitely generated cone. By Weyl’s Theorem, $K$ is also finitely constrained, so $K = \{ x \in \mathbb{R}^n : Bx \geq 0 \}$ for some matrix $B$. Again from Corollary 2, we have $L = K^+ = \{ B^T z : z \geq 0 \}$, so $L$ is indeed finitely generated.
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Example of homogenization

The proof of the affine Weyl Theorem uses a technique called **homogenization**, where a polyhedron is embedded in a higher dimensional cone.

**Example**

The polyhedron
\[ \{ x_1 \in \mathbb{R}^1 : x_1 \geq 3 \} = \{ x_1 \in \mathbb{R}^1 : x_1 = 5y + 3z, y \geq 0, z \geq 0, z = 1 \} \]
can be regarded as the slice of the cone
\[ \{ x \in \mathbb{R}^2 : x_1 = 5y + 3z, x_2 = z, y, z \geq 0 \} \]
\[ = \{ x \in \mathbb{R}^2 : x_1 - 3x_2 \geq 0, x_2 \geq 0 \} \subseteq \mathbb{R}^2 \]
with \( x_2 = 1 \).

Note that the homogenized version is the closure of the conic hull of the slice corresponding to the polyhedron.
Affine Weyl

Theorem (Affine Weyl Theorem)

Finitely generated polyhedra are finitely constrained: Let

\[
P := \{ x \in \mathbb{R}^n : x = By + Cz, \ y \geq 0, \ z \geq 0 \ \sum_{i=1}^{q} z_i = 1 \}
\]

with \( B \in \mathbb{R}^{n \times p} \) and \( C \in \mathbb{R}^{n \times q} \). Then there exists a matrix \( A \in \mathbb{R}^{m \times n} \) and a vector \( b \in \mathbb{R}^m \) such that

\[
P = \{ x \in \mathbb{R}^n : Ax \geq b \}.
\]
Proof of Affine Weyl

If \( P = \emptyset \), we can take \( m = 1 \), \( A \) to be composed of zeroes, and \( b = 1 \). Otherwise, if \( P \neq \emptyset \):
Define

\[
P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & e^T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, y \geq 0, z \geq 0 \right\}
\]

where \( e \in \mathbb{R}^q \) is a vector of ones. Note that \( P \) corresponds to points in \( P' \) with \( x_{n+1} = 1 \). The set \( P' \) is a finitely generated cone, so by Weyl’s Theorem it is also finitely constrained:

\[
P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : [A : -b] \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geq 0 \right\}
\]
Proof of Affine Weyl 2

Then

\[
P \; = \; \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix} \in P' \right\}
\]

\[
= \left\{ x : \left[ A^T - b \right] \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \right\}
\]

\[
= \left\{ x : Ax \geq b \right\}
\]

as required. QED
Affine Minkowski

Theorem (Affine Minkowski Theorem)

Finitely constrained polyhedra are finitely generated:
Suppose \( P = \{ x \in \mathbb{R}^n : Ax \geq b \} \) where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Then there exist matrices \( B \in \mathbb{R}^{n \times p} \) and \( C \in \mathbb{R}^{n \times q} \) such that

\[
P = \{ x \in \mathbb{R}^n : x = By + Cz, \ y \geq 0, \ z \geq 0 \sum_{i=1}^{q} z_i = 1 \}.
\]
Proof of Affine Minkowski

This is also proved using homogenization. First take care of the trivial case: If $P = \emptyset$, can take $p = q = 0$, so $B$ and $C$ are vacuous. Now the general case, with $P \neq \emptyset$. Let

$$P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} A & -b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geq 0 \right\}$$

so again $x \in P \iff \begin{bmatrix} x \\ 1 \end{bmatrix} \in P'$. By Minkowski’s Theorem, we have

$$P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = Dw, w \geq 0 \right\}$$

for some $D \in \mathbb{R}^{(n+1) \times m}$. 
Proof of Affine Minkowski 2

We can rearrange the columns of $D$ if necessary so that

$$P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix} \neq 0 \ldots \neq 0 \right\} \begin{bmatrix} y \\ z \end{bmatrix}, \ y, z \geq 0 \right\}. $$

In order to get the desired form, we need to pin down the nonzeroes in the last row of $D$. Note that if any of the nonzeroes are negative then we could choose the corresponding $z_j > 0$ and $y$ and all other components of $z$ equal to zero. This would give a point with $x_{n+1} < 0$, which is not in $P'$. So all the nonzeroes in the last row must be positive.
Proof of Affine Minkowski 3

By scaling the last $q$ columns of $D$, we can rewrite as

$$P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & e^T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, \ y \geq 0, \ z \geq 0 \right\}$$

and

$$P = \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix} \in P' \right\}$$

$$= \left\{ x \in \mathbb{R}^n : x = By + Cz, \ y \geq 0, \ z \geq 0 \sum_{i=1}^{q} z_i = 1 \right\}$$

as desired. QED
Goldman Resolution Theorem

Together, the affine Weyl and the affine Minkowski Theorems are known as the Double Decomposition Theorem. They can be extended to the following theorem:

Theorem (Goldman Resolution Theorem)

Suppose \( P = \{ x \in \mathbb{R}^n : Ax \geq b \} \neq \emptyset \) where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Then \( P = S + K + Q \), where

\[
\begin{align*}
S &= \{ x \in \mathbb{R}^n : Ax = 0 \} \\
S + K &= \{ x \in \mathbb{R}^n : Ax \geq 0 \} \\
K &= \text{a pointed cone} \\
K + Q &= \text{a pointed polyhedron} \\
Q &= \text{a polytope given by the convex hull of extreme points of } K + Q
\end{align*}
\]
Example

An example where $S \neq \{0\}$ and $K$ and $Q$ are not unique:

$$P = \{ x \in \mathbb{R}^2 : x_1 + x_2 \geq 3 \}$$

$$= \left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, t \geq 0 \right\}$$

$$= \left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \end{bmatrix}, t \geq 0 \right\}$$

For example,

$$x = \begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-7) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 10 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -2 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$