Expressing MaxCut as a Semidefinite Program

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Let $G = (V, E)$ be a graph with edge weights $w_e$. Let $n = |V|$. For any subset $U \subseteq V$, let $\delta(U)$ denote the edges in $E$ with exactly one endpoint in $U$ and let $E(U)$ denote the set of edges with both endpoints in $U$. Given a partition of $V$ into $V_1$ and $V_2$, the value of the corresponding cut can be expressed in two ways as

$$z(V_1, V_2) = \sum_{e \in \delta(V_1)} w_e - \sum_{e \in E(V_1)} w_e - \sum_{e \in E(V_2)} w_e.$$

For notational convenience, for any missing edge $(u, v) \in (V \times V) \setminus E$, we define $w_{uv} = 0$. Combining the two formulations for MaxCut, we also have

$$z(V_1, V_2) = 0.5 \left( \sum_{e \in E} w_e - \sum_{e \in E(V_1)} w_e - \sum_{e \in E(V_2)} w_e + \sum_{e \in \delta(V_1)} w_e \right).$$

This can be expressed more concisely in terms of the Laplacian matrix of the weighted graph:

$$L_G = D_G - W_G,$$

where the entries of $W_G$ are the edge weights $w_e$ and where $D_G$ is a diagonal matrix with

$$D_G(i, i) = \sum_{j \in V} w_{ij}.$$

We then have

$$z(V_1, V_2) = 0.25 x^T L_G x$$

with $x_i = 1$ if $i \in V_1$ and $x_i = -1$ if $i \in V_2$. Hence, the MaxCut problem can be written as the quadratic binary problem

$$\begin{align*}
\max_x & \quad 0.25 x^T L_G x \\
\text{subject to} & \quad x_i = \pm 1 \quad \forall i \in V.
\end{align*}$$

(1)
This problem can then be relaxed to a semidefinite program. First, note from properties of the trace function that

\[ x^T L_G x = \text{trace}(x^T L_G x) = \text{trace}(L_G xx^T) \]

Now, introduce an \( n \times n \) matrix \( X \). We can express problem (1) equivalently as

\[
\begin{align*}
\max_{x, X} & \quad 0.25 \text{trace}(L_G X) \\
\text{subject to} & \quad x_i = \pm 1 \quad \forall i \in V \\
& \quad X = xx^T
\end{align*}
\]

which in turn is equivalent to the problem

\[
\begin{align*}
\max_{x, X} & \quad 0.25 \text{trace}(L_G X) \\
\text{subject to} & \quad X_{ii} = 1 \quad \text{for } i = 1, \ldots, n \\
& \quad X = xx^T
\end{align*}
\]

Since \( X = xx^T \), we must have that \( X \) is symmetric and positive semidefinite. Furthermore, it must have rank equal to 1. Relaxing the restriction on the rank, we get the following semidefinite programming relaxation of MaxCut:

\[
\begin{align*}
\max_X & \quad 0.25 \text{trace}(L_G X) \\
\text{subject to} & \quad X_{ii} = 1 \quad \text{for } i = 1, \ldots, n \\
& \quad X \succeq 0, \quad (2)
\end{align*}
\]

where the notation \( X \succeq 0 \) is equivalent to the requirement that \( X \) be symmetric and positive semidefinite.

Goemans and Williamson [1] showed that any optimal solution to (2) can be rounded to a feasible solution to MaxCut with value at least 0.878 of the optimal value of MaxCut, provided all the edge weights \( w_e \) are nonnegative. Nesterov [2] showed the corresponding ratio is \( \frac{2}{\pi} \) for general edge weights.

References
