We want to solve the linear program

\[
\begin{array}{rl}
\min_{x \in \mathbb{R}^n} & c^T x \\
\text{subject to} & Ax = b \\
& Hx = h \\
& x \geq 0
\end{array}
\]  \tag{1}

where \( A \in \mathbb{R}^{m_A \times n} \) and \( H \in \mathbb{R}^{m_H \times n} \), with all vectors dimensioned appropriately. Let

\[ X : \{ x \in \mathbb{R}^n : Hx = h, x \geq 0 \} \]

have extreme points \( \{ x^j : j \in J \} \) and extreme rays \( \{ d^k : k \in K \} \). In Dantzig-Wolfe decomposition, we work with the **Master Problem**:

\[
\begin{array}{rl}
\min_{\lambda, \mu} & \sum_{j \in J} (c^T x^j) \lambda_j + \sum_{k \in K} (c^T d^k) \mu_k \\
\text{subject to} & \sum_{j \in J} (Ax^j) \lambda_j + \sum_{k \in K} (Ad^k) \mu_k = b \\
& \sum_{j \in J} \lambda_j = 1 \\
& \lambda, \mu \geq 0
\end{array}
\]  \tag{2}

The **Master Dual** is then

\[
\begin{array}{rl}
\max_{\pi \in \mathbb{R}^{m_A}, \sigma \in \mathbb{R}} & b^T \pi + \sigma \\
\text{subject to} & (c - A^T \pi)^T x^j \geq \sigma \quad \text{for all } j \in J \\
& (c - A^T \pi)^T d^k \geq 0 \quad \text{for all } k \in K
\end{array}
\]  \tag{3}

The Master Problem can be solved using the revised simplex method, giving a dual solution \( \bar{\pi}, \bar{\sigma} \) at each iteration. In order to check dual feasibility, we solve the **subproblem**

\[
\begin{array}{rl}
\min_{x \in \mathbb{R}^n} & (c - A^T \bar{\pi})^T x \\
\text{subject to} & Hx = h \\
& x \geq 0
\end{array}
\]  \tag{4}

If the current solution to \( (2) \) is not optimal then the solution to the subproblem gives an entering column with a negative reduced cost.

An alternative approach to solve \( (1) \) is to use Lagrangian Relaxation. We can set up a Lagrangian relaxation for any \( \zeta \in \mathbb{R}^{m_A} \):

\[
z(\zeta) := \min_{x \in \mathbb{R}^n} \quad c^T x + \zeta^T (b - Ax) \\
\text{subject to} \quad Hx = h \\
x \geq 0
\]  \tag{5}
This gives a lower bound on the optimal value of (1) for any $\zeta$. The Lagrangian dual problem is to maximize $z(\zeta)$:

$$\max_{\zeta \in \mathbb{R}^m} z(\zeta).$$

(6)

Equivalently, we obtain

$$\max_{\zeta \in \mathbb{R}^m} \left( b^T \zeta + \min_{x \in \mathbb{R}^n} \{(c - A^T \zeta)^T x : H x = h, x \geq 0\} \right),$$

which in turn is equivalent to the problem

$$\max_{\zeta \in \mathbb{R}^m, \nu \in \mathbb{R}} b^T \zeta + \nu \quad \text{subject to} \quad (c - A^T \zeta)^T x \geq \nu \quad \forall x \in X.$$  

(7)

This can be expressed in an equivalent form using the extreme points and extreme rays of $X$, which then gives exactly the Master Dual, under the correspondence $\pi \leftrightarrow \zeta$ and $\sigma \leftrightarrow \nu$.

The Dantzig-Wolfe column generation approach to solve (1) translates into a cutting plane method to solve the Lagrangian dual problem (7).