We work with the standard form linear program and its dual:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax = b \quad (P) \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \quad (D)
\end{align*}
\]

where \( c \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \), and \( A \in \mathbb{R}^{m \times n} \). The results extend to any primal-dual pair, including

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \quad (\hat{P}) \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \\
& \quad y \geq 0
\end{align*}
\]

**Theorem 1. Weak duality:** If \( \bar{x} \) is feasible in \( (P) \) and \( \bar{y} \) is feasible in \( (D) \) then \( c^T x \geq b^T y \).

**Proof.** We have

\[
b^T \bar{y} = (A\bar{x})^T \bar{y} = \bar{x}^T A^T \bar{y} \leq \bar{x}^T c = c^T \bar{x}
\]

where the inequality follows because \( \bar{x} \geq 0 \) and \( A^T \bar{y} \leq c \). \( \square \)

As an immediate consequence of the weak duality theorem, we have the following sufficient condition for optimality:

**Theorem 2. Sufficient optimality criterion:** If \( \bar{x} \) is primal feasible and \( \bar{y} \) is dual feasible and \( c^T \bar{x} = b^T \bar{y} \) then \( \bar{x} \) is optimal in \( (P) \) and \( \bar{y} \) is optimal in \( (D) \).

This condition is also necessary:

**Theorem 3. Strong duality theorem:** If \( (P) \) is feasible with a finite optimal value then \( (D) \) is also feasible. Further, there exist optimal solutions \( x^* \) and \( y^* \) with \( c^T x^* = b^T y^* \).

We will prove this theorem later. In particular, we will use the simplex algorithm to show how a dual optimal solution can be constructed with \( c^T x^* = b^T y^* \).

A general optimization problem can be in one of four possible states:

- Infeasible
- Feasible with an unbounded optimal value.
- Feasible with a finite optimal value that is attained.
- Feasible with a finite optimal value that is not attained.

It follows from the strong duality theorem that this last case is not possible for a linear program. The strong duality theorem also implies that the state of a linear program tells us the state of its dual linear program. So a primal-dual pair can only be in one of 4 possible states, rather than \( 4 \times 4 = 16 \) possible states. In particular, we have the following consequence of the strong duality theorem:
Theorem 4. The following are mutually exclusive and exhaustive possibilities for \((P)\) and \((D)\):

- \((P)\) and \((D)\) are both infeasible.
- \((D)\) is infeasible and \((P)\) has an unbounded optimal value.
- \((P)\) is infeasible and \((D)\) has an unbounded optimal value.
- Both \((P)\) and \((D)\) are feasible, and they have the same optimal value.

Complementary slackness

Theorem 5. A pair of primal and dual feasible solutions are optimal to their respective problems in a primal-dual pair of LPs if and only if

\[
\text{complementary slackness}
\]

whenever these variables make a slack variable in one problem strictly positive the value of the associated nonnegative variable in the other is zero.

Proof. We prove this for the standard pair \((P)\) and \((D)\). We can define the vector of dual slacks \(s = c - A^T y \in \mathbb{R}^n\) for any \(y \in \mathbb{R}^m\). Note that the duality gap is

\[
c^T x - b^T y = c^T x - (Ax)^T y = c^T x - x^T A^T y = c^T x - (A^T y)^T x = (c - A^T y)^T x = s^T x = \sum_{i=1}^n s_i x_i
\]

for any \(y \in \mathbb{R}^m\) and \(x \in \mathbb{R}^n\) with \(Ax = b\). Note that if \(x\) and \(y\) are feasible in their respective problems then \(x \geq 0\) and \(s \geq 0\), so \(s^T x \geq 0\).

If the points are optimal then \(c^T x - b^T y = 0\) so \(\sum_{i=1}^n s_i x_i = 0\), so each component \(s_i x_i = 0\), since they must all be nonnegative. So either \(s_i = 0\) or \(x_i = 0\) for each component \(i\). This is complementary slackness.

If complementary slackness holds then either \(x_i = 0\) or \(s_i = 0\) for each component \(i\), so \(s^T x = 0\) so \(c^T x = b^T y\) so the points are optimal. \(\square\)

Exercise: Consider the primal-dual pair

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \quad (P) \\
x & \geq 0
\end{align*}
\]

\[
\begin{align*}
\max_y & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \quad (D) \\
y & \geq 0
\end{align*}
\]

and let \(s = c - A^T y\), \(w = Ax - b\). Show \(c^T x - b^T y = x^T s + y^T w\).

Theorem 6. Farkas Lemma: Let \(A \in \mathbb{R}^{m \times n}\) and \(b \in \mathbb{R}^m\). Exactly one of the following two systems has a solution:

\[
(\text{I}): \quad Ax = b, \quad x \geq 0. \\
(\text{II}): \quad A^T y \leq 0, \quad b^T y > 0.
\]

Proof. Prove using LP duality. Set up the primal-dual pair \((P)\) and \((D)\) with \(c = 0\). Note that \(y = 0\) is feasible in \((D)\), so \((D)\) either has finite optimal value or unbounded optimal value. Note also that \((P)\) is either infeasible or has optimal value equal to zero.

If \((\text{I})\) is consistent then \((P)\) has optimal value 0 so \((D)\) also has optimal value 0, so there is no \(y \in \mathbb{R}^m\) with both \(A^T y \leq 0\) and \(b^T y > 0\), so \((\text{II})\) is inconsistent.

If \((\text{I})\) is inconsistent then \((P)\) is infeasible so \((D)\) has unbounded optimal value, so there exists \(\bar{y}\) with \(A^T \bar{y} \leq 0\) and \(b^T \bar{y} > 0\), so \((\text{II})\) is consistent. \(\square\)