There are two fundamentally different ways to look at polyhedra:

- **As constrained polyhedra:** $P = \{x \in \mathbb{R}^n : Ax \geq b\}$.
- **As finitely generated polyhedra:** $Q = \{x \in \mathbb{R}^n : x = Dy, \text{ with possible additional linear constraints on } y\}$.
  
  The columns of $D$ are the *generators* of $Q$.

The Weyl and Minkowski Theorems show that any polyhedron that is represented in one form can also be represented in the other form. The theorems are first stated and proved for the case of convex cones and then the results are extended out to more general polyhedra.

**Example 1.**

$$\left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} 4 & 2 & 5 & 3 & 1 \\ 1 & 2 & 1 & 2 & 8 \end{bmatrix} y, y \in \mathbb{R}^5, y \geq 0, y_3 + y_4 + y_5 = 1 \right\}$$

$$= \left\{ x \in \mathbb{R}^2 : x_1 + 2x_2 \geq 7, 3x_1 + x_2 \geq 11, -x_1 + 4x_2 \geq -1, 2x_1 - 2x_2 \geq -14 \right\}$$

Constrained and finitely generated convex cones have the forms:

- Constrained: $K = \{x \in \mathbb{R}^n : Ax \geq 0\}$ for some matrix $A$.
- Finitely generated: $K = \{x \in \mathbb{R}^n : x = Dy \text{ for some } y \geq 0\}$.

Recall that the dual of a set $M \subseteq \mathbb{R}^n$ is

$$M^+ := \{y \in \mathbb{R}^n : y^T x \geq 0 \forall x \in M\}.$$ 

**Example 2.** Find dual cones to (i) $\mathbb{R}^n$, (ii) $\mathbb{R}^n_+$, and (iii) the cone of symmetric positive semidefinite matrices.

The next proposition is left as an exercise:

**Proposition 1.** Let $S \subseteq \mathbb{R}^n$. We have $S = S^{++}$ if and only if $S$ is a constrained cone.

**Theorem 1** (Weyl). Any finitely generated convex cone is finitely constrained.

**Proof.** Let $K$ be a finitely generated cone. Then for some matrix $D$ we have

$$K = \{x \in \mathbb{R}^n : x = Dy \text{ for some } y \geq 0\}$$

$$= \{x \in \mathbb{R}^n : \text{the system } x - Dy = 0, y \geq 0 \text{ is consistent in } x, y\}$$

$$= \{x \in \mathbb{R}^n : Bx \geq 0\} \text{ for some matrix } B,$$

by using Fourier-Motzkin elimination to eliminate $y$. \qed
Corollary 1. Given $A \in \mathbb{R}^{m \times n}$, let $K := \{Ax : x \geq 0\} \subseteq \mathbb{R}^m$ and $L := \{y \in \mathbb{R}^m : A^Ty \geq 0\}$. Then $K^+ = L$ and $L^+ = K$.

Proof. We first show $K^+ = L$. We have

$K^+ = \{z \in \mathbb{R}^m : z^Tw \geq 0 \forall w \in K\}$

$= \{z \in \mathbb{R}^m : z^TAx \geq 0 \forall x \geq 0\}$

$= \{z \in \mathbb{R}^m : (A^Tz)^T x \geq 0 \forall x \geq 0\}$

which holds if and only if $A^Tz \geq 0$, so $K^+ = L$.

Now we show $L^+ = K$. By Weyl’s Theorem, the cone $K$ can also be expressed as a constrained cone, so there is a matrix $C$ with $K = \{x \in \mathbb{R}^n :Cx \geq 0\}$. From Proposition 1, we have $K = K^{++} = (K^+)^+ = L^+$, as required.

Corollary 2 (Farkas Lemma). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following systems has a solution:

(I) $Ax = b, x \geq 0$.

(II) $A^Ty \leq 0, b^Ty > 0$.

Proof. Define $K$ and $L$ as in Corollary 1.

First, assume (I) fails. Then $b \not\in K$. By Weyl’s Theorem and Proposition 1, we have $b \not\in K^{++} = (K^+)^+$. So there exists $y \in K$ with $b^Ty < 0$. From Corollary 1, we have $K^+ = \{y \in \mathbb{R}^m : A^Ty \geq 0\}$, so (II) is consistent.

Now, assume (II) holds, so $\exists y \in L$ with $b^Ty < 0$. So $b \not\in L^+ = K$ by Corollary 1, so (I) fails.

Corollary 3 (Minkowski’s Theorem). Any finitely constrained cone is nonempty and finitely generated.

Proof. Define $K$ and $L$ as in Corollary 1. We need to show that the generic finitely constrained cone $L$ is finitely constrained and nonempty. Note first that the origin is a point in $L$, so $L$ is nonempty.

Now we need to show that $L$ is finitely generated. We know from Corollary 1 that $L^+ = K$, a finitely generated cone. By Weyl’s Theorem, $K$ is also finitely constrained, so $K = \{x \in \mathbb{R}^n : Bx \geq 0\}$ for some matrix $B$. Again from Corollary 1, we have $L = K^+ = \{B^Tz : z \geq 0\}$, so $L$ is indeed finitely generated.

The proof of the affine Weyl Theorem uses a technique called homogenization, where a polyhedron is embedded in a higher dimensional cone.

Example 3. The polyhedron $\{x_1 \in \mathbb{R}^1 : x_1 \geq 3\} = \{x_1 \in \mathbb{R}^1 : x_1 = 5y + 3z, y \geq 0, z \geq 0, z = 1\}$ can be regarded as the slice of the cone

$\{x \in \mathbb{R}^2 : x_1 = 5y + 3z, x_2 = z, y, z \geq 0\} = \{x \in \mathbb{R}^2 : x_1 - 3x_2 \geq 0, x_2 \geq 0\} \subseteq \mathbb{R}^2$

with $x_2 = 1$. The homogenized version is the closure of the conic hull of the slice corresponding to the polyhedron.
Theorem 2 (Affine Weyl Theorem). Finitely generated polyhedra are finitely constrained:

Let

\[ P := \{ x \in \mathbb{R}^n : x = By + Cz, y \geq 0, z \geq 0, \sum_{i=1}^{q} z_i = 1 \} \]

with \( B \in \mathbb{R}^{n \times p} \) and \( C \in \mathbb{R}^{n \times q} \). Then there exists a matrix \( A \in \mathbb{R}^{m \times n} \) and a vector \( b \in \mathbb{R}^m \) such that

\[ P = \{ x \in \mathbb{R}^n : Ax \geq b \}. \]

Proof. If \( P = \emptyset \), we can take \( m = 1, A \) to be composed of zeroes, and \( b = 1 \).

Otherwise, if \( P \neq \emptyset \):

Define

\[
P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, y \geq 0, z \geq 0 \right\}
\]

where \( e \in \mathbb{R}^q \) is a vector of ones. Note that \( P \) corresponds to points in \( P' \) with \( x_{n+1} = 1 \).

The set \( P' \) is a finitely generated cone, so by Weyl’s Theorem it is also finitely constrained:

\[
P' = \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : [A : -b] \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geq 0 \right\}
\]

Then

\[
P = \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix} \in P' \right\}
= \left\{ x : [A : -b] \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \right\}
= \left\{ x : Ax \geq b \right\}
\]

as required. \( \square \)

Theorem 3 (Affine Minkowski Theorem). Finitely constrained polyhedra are finitely generated:

Suppose \( P = \{ x \in \mathbb{R}^n : Ax \geq b \} \) where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Then there exist matrices \( B \in \mathbb{R}^{n \times p} \) and \( C \in \mathbb{R}^{n \times q} \) such that

\[ P = \{ x \in \mathbb{R}^n : x = By + Cz, y \geq 0, z \geq 0, \sum_{i=1}^{q} z_i = 1 \}. \]

Proof. This is also proved using homogenization.

First take care of the trivial case: If \( P = \emptyset \), can take \( p = q = 0 \), so \( B \) and \( C \) are vacuous.

Now the general case, with \( P \neq \emptyset \). Let

\[
P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} A & -b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \geq 0 \right\}
\]
so again \( x \in P \iff \begin{bmatrix} x \\ 1 \end{bmatrix} \in P' \). By Minkowski’s Theorem, we have

\[
P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = Dw, w \geq 0 \right\}
\]

for some \( D \in \mathbb{R}^{(n+1) \times m} \). We can rearrange the columns of \( D \) if necessary so that

\[
P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & e^T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, y \geq 0, z \geq 0 \right\}
\]

In order to get the desired form, we need to pin down the nonzeroes in the last row of \( D \). Note that if any of the nonzeroes are negative then we could choose the corresponding \( z_j > 0 \) and \( y \) and all other components of \( z \) equal to zero. This would give a point with \( x_{n+1} < 0 \), which is not in \( P' \). So all the nonzeroes in the last row must be positive. By scaling the last \( q \) columns of \( D \), we can rewrite as

\[
P' := \left\{ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} : \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & e^T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}, y \geq 0, z \geq 0 \right\}
\]

and

\[
P = \left\{ x : \begin{bmatrix} x \\ 1 \end{bmatrix} \in P' \right\} = \left\{ x \in \mathbb{R}^n : x = By + Cz, y \geq 0, z \geq 0, \sum_{i=1}^q z_i = 1 \right\}
\]

as desired.

Together, the affine Weyl and the affine Minkowski Theorems are known as the Double Decomposition Theorem. They can be extended to the following theorem, which states that any point \( x \in P \) can be represented as \( x = s + k + q \) where \( s \in S \) (the lineality space of \( P \)), \( k \in K \), and \( q \in Q \):

**Theorem 4 (Goldman Resolution Theorem).** Suppose \( P = \{ x \in \mathbb{R}^n : Ax \geq b \} \neq \emptyset \) where \( A \in \mathbb{R}^{m \times n} \) and \( b \in \mathbb{R}^m \). Then \( P = S + K + Q \), where

- \( S = \{ x \in \mathbb{R}^n : Ax = 0 \} \)
- \( S + K = \{ x \in \mathbb{R}^n : Ax \geq 0 \} \)
- \( K \) is a pointed cone
- \( K + Q \) is a pointed polyhedron
- \( Q \) is a polytope given by the convex hull of extreme points of \( K + Q \)

**Example 4.** An example where \( S \neq \{0\} \) and \( K \) and \( Q \) are not unique:

\[
P = \{ x \in \mathbb{R}^2 : x_1 + x_2 \geq 3 \} = \left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, t \geq 0 \right\} = \left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \end{bmatrix}, t \geq 0 \right\} = \ldots
\]