Perfect Graphs

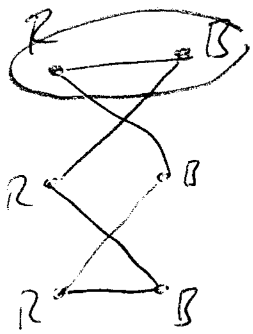
(Chudnovsky et al, Math Prog, 97(1-2), 2003, pp 405-422)

DEFN A graph $G=(V,E)$ is PERFECT if $\chi(\bar{G})=\omega(\bar{G})$ for every induced subgraph $\bar{G}$ of G,

where $\chi(G)$ = CHROMATIC NUMBER of G
= least number of colors needed to color the vertices

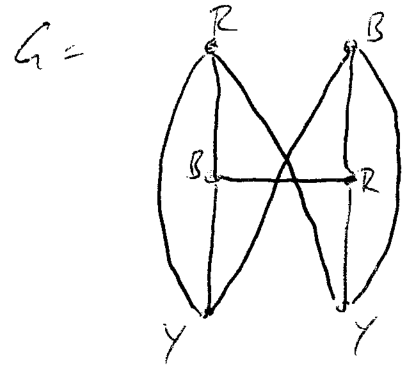
and $\omega(G)$ = the size of a maximum clique of G.

Eg: $G =$



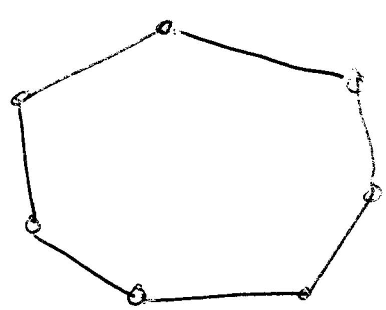
(bipartite graphs are perfect)

$G =$



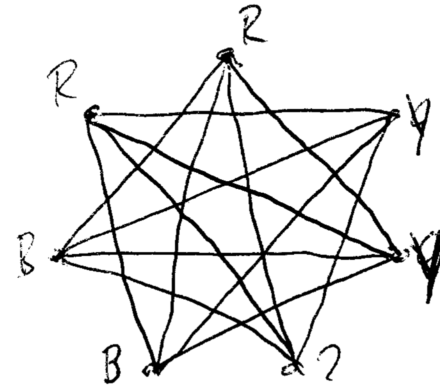
(complements of bipartite graphs are perfect)

Eg:



$\chi(G)=3$, $\omega(G)=2$

ODD-HOLES are not perfect



$\omega(G)=3$

ODD ANTIHOLES are not perfect.

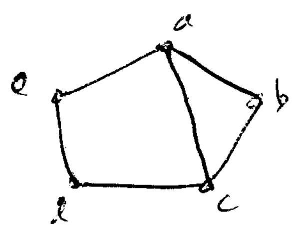
Packing LP:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq 1 \\ & x \geq 0 \end{aligned}$$

Every entry in A is 0 or 1.

Let rows of A correspond to incidence vectors of maximal cliques of a graph.

Eg:



$$A = \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

THM (Chvatal) The LP has an integral optimal solution for every objective c if and only if the underlying ~~the rows of A~~ graph is perfect.

Cliques in complement of G $\longleftrightarrow$ primal packing solutions

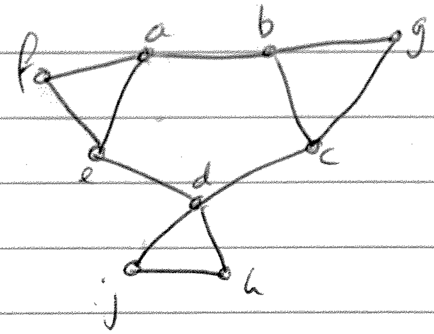
Colorings in complement of G $\longleftrightarrow$ dual solutions to packing LP.

THM A graph is perfect if and only if it has no odd hole and no odd antihole.

(Perfect graph conjecture of Berge (1961). Proved by Chudnovsky et al (2003).)

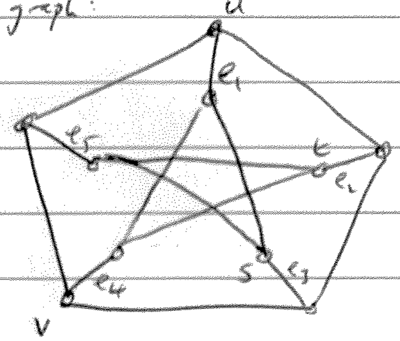
Example

(1)



This is not perfect:
it contains an odd hole.
Nevertheless, the size of the maximum node packing is equal to the cardinality of the minimum clique cover, namely 3.
Equality doesn't hold for the subgraph induced by $\{a,b,c,d,e\}$

(2) Petersen graph:



Not perfect – has odd holes.
Maximum node packing has cardinality 4 – $\{u, v, s, t\}$
Minimum clique cover has cardinality 5 – $\{e_1, e_2, e_3, e_4, e_5\}$
– there are no cliques of size 3 or larger.

(3) Illustrate Chvatal's result with graph in example (1):

$$
\begin{array}{c}
\begin{array}{ccccccccc} a & b & c & d & e & f & g & h & j \end{array} \\
A = \begin{bmatrix}
1 & 1 & & & & & & & \\
 & 1 & 1 & & & & 1 & & \\
 & & 1 & 1 & & & & & \\
 & & & 1 & & & & 1 & 1 \\
 & & & 1 & 1 & & & & \\
1 & & & & 1 & 1 & & &
\end{bmatrix}
\end{array}
$$

With c = vector of ones, get integral soln.
With $c = [1,1,1,1,1,0,0,0,0]$, get fractional soln:
$x_a = x_b = x_c = x_d = x_e = \frac{1}{2}$