Alternatives to Expectation

John E. Mitchell

1 Introduction

We will focus on stochastic two stage mixed integer programs with recourse. The general formulation can be written

$$\begin{align*}
\min_x & \quad c^T x + E_{\xi} [Q(x, \xi)] \\
\text{subject to} & \quad A x = b \\
& \quad x \in \mathcal{X}
\end{align*}$$

(1)

where the first stage decisions are $x \in \mathcal{X} \subseteq \mathbb{R}^{n_1}$, the constraint matrix $A$ is $m_1 \times n_1$, $b \in \mathbb{R}^{m_1}$, $c \in \mathbb{R}^{n_1}$, $\xi$ is the uncertainty, and $Q(x, \xi)$ is the cost of the recourse decision when the first stage decision is $x$ and the uncertainty is $\xi$. Thus, $Q(x, \xi)$ is the second stage cost. We take the expectation of the second stage cost over all scenarios $\xi$.

The second stage cost

$$Q(x, \xi) = \min_y q^T y$$

subject to

$$W y = h(\xi) - T(\xi) x$$

where $y \in \mathcal{Y} \subseteq \mathbb{R}^{n_2}$, $W$ is a fixed $m_2 \times n_2$ matrix, the right hand side $h(\xi) \in \mathbb{R}^{m_2}$ depends on the uncertainty $\xi$, and the $m_2 \times n_2$ technology matrix $T(\xi)$ also depends on $\xi$. Note that the second stage optimization is over $y$, with $x$ taken as a parameter.

The sets $\mathcal{X}$ and $\mathcal{Y}$ impose nonnegativity, and discrete, binary, or continuous restrictions on the first and second-stage variables, respectively.

Sometimes the expectation does not capture the risk sufficiently. There are several alternatives.

2 Robust optimization

Replace expectation by worst possible outcome:

$$\begin{align*}
\min_{x, v} & \quad c^T x + v \\
\text{subject to} & \quad A x = b \\
& \quad v \geq Q(x, \xi) \quad \forall \xi \\
& \quad x \in \mathcal{X}
\end{align*}$$

(2)

This doesn’t require knowledge of the density function for $\xi$. This is conservative.

A slightly less conservative choice is to only consider $\xi$ within some set:

$$\begin{align*}
\min_x & \quad c^T x + v \\
\text{subject to} & \quad A x = b \\
& \quad v \geq Q(x, \xi) \quad \forall \xi \in \Xi \\
& \quad x \in \mathcal{X}
\end{align*}$$

(3)

Typical choices for $\Xi$ are ellipsoids or boxes.
3 Chance constrained formulations

Require that $Q(x, \xi)$ be below some threshold $\beta$ with probability $1 - \alpha$:

$$\begin{align*}
\min_x & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad \Pr(Q(x, \xi) > \beta) \leq \alpha \\
& \quad x \in X
\end{align*}$$

(A4)

A special case is minimizing the Value-at-Risk (VaR):

$$\begin{align*}
\min_{x, \eta} & \quad \eta \\
\text{subject to} & \quad Ax = b \\
& \quad \Pr(Q(x, \xi) > \eta) \leq \alpha \\
& \quad x \in X
\end{align*}$$

(A5)

With $\alpha = 0.1$, this is choosing $x$ so that 90th percentile outcome is as good as possible.

When the uncertainty $\xi$ consists of a finite number of scenarios, these problems can be modeled as equivalent integer programs.

4 Conditional Value-at-Risk (CVaR)

Instead of taking the expectation of all possible outcomes, take the expectation of the worst $\alpha$ outcomes. Perhaps, take the expectation of the worst 10% of outcomes.

$$\begin{align*}
\min_x & \quad c^T x + \mathbb{E}_{\xi} [Q(x, \xi) \mid Q(x, \xi) \text{ is in worst } \alpha\% \text{ of outcomes}] \\
\text{subject to} & \quad Ax = b \\
& \quad x \in X
\end{align*}$$

(A6)

With a finite set of scenarios, this can be modeled as a mixed integer program:

$$\begin{align*}
\min_{x, v, \eta} & \quad c^T x + \eta + \frac{1}{\alpha} \sum_s p_s v_s \\
\text{subject to} & \quad Ax = b \\
& \quad Q(x, \xi_s) - \eta \leq v_s \quad \forall s \\
& \quad v_s \geq 0 \quad \forall s \\
& \quad x \in X
\end{align*}$$

(A7)

This MIP is only slightly more complicated than the original stochastic program (A1). The values $v_s$ give the excess of $Q(x, \xi_s)$ over $\eta$. The value of $\eta$ can fall in a range: anywhere between the best $(1 - \alpha)$ outcomes and the worst $\alpha$ outcomes.

5 Coherent risk measures

Can combine minimizing the expected value with minimizing some measure of risk.

There is a theory of coherent risk measures which possess various nice properties:

- convexity, monotonicity, translation equivalence, positive homogeneity

**Variance not a coherent risk measure.** It fails monotonicity, which requires:

- if the outcome is always worse then the risk measure should be higher

CVaR is coherent.

VaR is not coherent: it fails convexity.