MATP6620/ISYE6760 Integer and Combinatorial Optimization

Alternatives to Expectation

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1 Introduction

We will focus on **stochastic two stage mixed integer programs with recourse**. The general formulation can be written

$$\begin{array}{lrcl} \min_x & c^T x & + & E_\xi\left[Q(x,\xi)\right] \\ \text{subject to} & Ax & = & b \\ & x & \in & \mathcal{X} \end{array} \tag{1}$$

where the first stage decisions are $x \in \mathcal{X} \subseteq \mathbb{R}^{n_1}$, the constraint matrix A is $m_1 \times n_1$, $b \in \mathbb{R}^{m_1}$, $c \in \mathbb{R}^{n_1}$, ξ is the uncertainty, and $Q(x,\xi)$ is the cost of the recourse decision when the first stage decision is x and the uncertainty is ξ. Thus, $Q(x,\xi)$ is the second stage cost. We take the expectation of the second stage cost over all scenarios ξ.

The second stage cost

$$\begin{array}{rlrcl} Q(x,\xi) = & \min_y & q^T y & & \\ & \text{subject to} & Wy & = & h(\xi) - T(\xi)x \\ & & y & \in & \mathcal{Y} \end{array}$$

where $y \in \mathcal{Y} \subseteq \mathbb{R}^{n_2}$, W is a fixed $m_2 \times n_2$ matrix, the right hand side $h(\xi) \in \mathbb{R}^{m_2}$ depends on the uncertainty ξ, and the $m_2 \times n_2$ technology matrix $T(\xi)$ also depends on ξ. Note that the second stage optimization is over y, with x taken as a parameter.

The sets $\mathcal{X}$ and $\mathcal{Y}$ impose nonnegativity, and discrete, binary, or continuous restrictions on the first and second-stage variables, respectively.

Sometimes the expectation does not capture the risk sufficiently. There are several alternatives.

2 Robust optimization

Replace expectation by worst possible outcome:

$$\begin{array}{lrcll} \min_{x,v} & c^T x & + & v & \\ \text{subject to} & Ax & = & b & \\ & v & \geq & Q(x,\xi) & \forall\, \xi \\ & x & \in & \mathcal{X} & \end{array} \tag{2}$$

This doesn't require knowledge of the density function for ξ. This is conservative.

A slightly less conservative choice is to only consider ξ within some set:

$$\begin{array}{lrcll} \min_x & c^T x & + & v & \\ \text{subject to} & Ax & = & b & \\ & v & \geq & Q(x,\xi) & \forall\, \xi \in \Xi \\ & x & \in & \mathcal{X} & \end{array} \tag{3}$$

Typical choices for Ξ are ellipsoids or boxes.

3 Chance constrained formulations

Require that $Q(x,\xi)$ be below some threshold β with probability $1-\alpha$:

$$\begin{array}{lrcl} \min_x & c^T x & & \\ \text{subject to} & Ax & = & b \\ & \Pr(Q(x,\xi) > \beta) & \leq & \alpha \\ & x & \in & \mathcal{X} \end{array} \tag{4}$$

A special case is minimizing the **Value-at-Risk (VaR)**:

$$\begin{array}{lrcl} \min_{x,\eta} & \eta & & \\ \text{subject to} & Ax & = & b \\ & \Pr(Q(x,\xi) > \eta) & \leq & \alpha \\ & x & \in & \mathcal{X} \end{array} \tag{5}$$

With $\alpha = 0.1$, this is choosing x so that 90th percentile outcome is as good as possible.

When the uncertainty ξ consists of a finite number of scenarios, these problems can be modeled as equivalent **integer programs**.

4 Conditional Value-at-Risk (CVaR)

Instead of taking the expectation of all possible outcomes, take the **expectation of the worst α outcomes**. Perhaps, take the expectation of the worst 10% of outcomes.

$$\begin{array}{lrcl} \min_x & c^T x & + & E_\xi\left[Q(x,\xi) \,|\, Q(x,\xi) \text{ is in worst } \alpha\% \text{ of outcomes}\right] \\ \text{subject to} & Ax & = & b \\ & x & \in & \mathcal{X} \end{array} \tag{6}$$

With a finite set of scenarios, this can be modeled as a mixed integer program:

$$\begin{array}{lrcrcl} \min_{x,v,\eta} & c^T x & + & \eta & + & \frac{1}{\alpha}\sum_s p_s v_s \\ \text{subject to} & Ax & & & = & b \\ & Q(x,\xi_s) & - & \eta & \leq & v_s \quad \forall\, s \\ & & & v_s & \geq & 0 \quad \forall\, s \\ & x & & & \in & \mathcal{X} \end{array} \tag{7}$$

This MIP is only slightly more complicated than the original stochastic program (1). The values v_s give the excess of $Q(x,\xi_s)$ over η. The value of η can fall in a range: anywhere between the best $(1-\alpha)$ outcomes and the worst α outcomes.

5 Coherent risk measures

Can combine minimizing the expected value with minimizing some measure of risk.

There is a theory of **coherent risk measures** which possess various nice properties:

convexity, monotonicity, translation equivalence, positive homogeneity

Variance not a coherent risk measure. It fails monotonicity, which requires:

if the outcome is always worse then the risk measure should be higher

CVaR is coherent.
VaR is not coherent: it fails convexity.