MATP6620/ISYE6760 Integer and Combinatorial Optimization

The L-Shaped Decomposition Method

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1 Introduction

We will focus on **stochastic two stage integer linear programs with recourse**. The general formulation can be written

$$\begin{array}{lrcl} \min_x & c^T x & + & \mathbb{E}_\xi\left[Q(x,\xi)\right] \\ \text{subject to} & Ax & = & b \\ & x & \in & \mathcal{X} \end{array}$$

where the first stage decisions are $x \in \mathcal{X} \subseteq \mathbb{R}^{n_1}$, the constraint matrix A is $m_1 \times n_1$, $b \in \mathbb{R}^{m_1}$, $c \in \mathbb{R}^{n_1}$, ξ is the uncertainty, and $Q(x,\xi)$ is the cost of the recourse decision when the first stage decision is x and the uncertainty is ξ. Thus, $Q(x,\xi)$ is the second stage cost. We take the expectation of the second stage cost over all scenarios ξ.

The second stage cost

$$\begin{array}{rlrcl} Q(x,\xi) = & \min_y & q^T y & & \\ & \text{subject to} & Wy & = & h(\xi) - T(\xi)x \\ & & y & \in & \mathcal{Y} \end{array}$$

where $y \in \mathcal{Y} \subseteq \mathbb{R}^{n_2}$, W is a fixed $m_2 \times n_2$ matrix, the right hand side $h(\xi) \in \mathbb{R}^{m_2}$ depends on the uncertainty ξ, and the $m_2 \times n_2$ technology matrix $T(\xi)$ also depends on ξ. Note that the second stage optimization is over y, with x taken as a parameter.

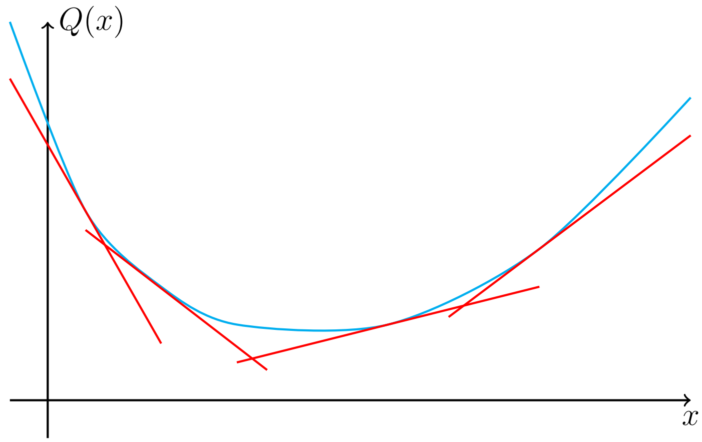
The sets $\mathcal{X}$ and $\mathcal{Y}$ impose nonnegativity, and discrete, binary, or continuous restrictions on the first and second-stage variables, respectively.

To simplify the presentation, we assume we have **complete fixed recourse**, so the second stage problem is feasible for any choice of x in the first stage. We also assume the number of scenarios is finite and thatthe second stage problem has bounded optimal value.

2 The L-shaped method for continuous second stage variables

Lemma 1. *If the second stage variables are all continuous, the second stage cost $Q(x,\xi)$ and the expectation $\mathbb{E}_\xi\left[Q(x,\xi)\right]$ are convex functions of x for any ξ.*

When $\mathbb{E}_\xi\left[Q(x,\xi)\right]$ is a convex function, we can approximate it using a piecewise linear function. The pieces can be generated as needed in a column generation algorithm. The resulting algorithm is known as the **L-Shaped decomposition method**.



2.1 Finding the subgradients

Assume we have a finite number of scenarios $s = 1, \ldots, S$, each with probability p_s. Then

$$Q(x) \;=\; \sum_{i=1}^{S} p_s Q(x, \xi^s). \tag{1}$$

In the Master Problem, we choose a first stage decision $\bar{x}$. We then want to find a valid constraint of the form

$$Q(x) \;\geq\; Q(\bar{x}) \;-\; \zeta^T(x - \bar{x}) \tag{2}$$

for some vector ζ. Such a valid constraint exists because $Q(x)$ is convex. This is a subgradient inequality.

By LP duality, for fixed $\bar{x}$

$$\begin{array}{rclrcl}
Q(\bar{x}, \xi^s) & = & \min_y & q^T y & & \\
 & & \text{subject to} & Wy & = & h(\xi^s) - T(\xi^s)\bar{x} \\
 & & & y & \geq & 0 \\
 & & & & & \\
 & = & \max_\pi & \multicolumn{3}{l}{\left(h(\xi^s) - T(\xi^s)\bar{x}\right)^T \pi} \\
 & & \text{subject to} & W^T \pi & \leq & q \\
 & & & \pi & & \text{free}
\end{array}$$

Let $\bar{\pi}^s$ be an optimal solution to the dual of the subproblem. Then $\bar{\pi}^s$ is feasible in the dual to the subproblem for any x, so we have the following inequality for $Q(x, \xi^s)$:

$$Q(x, \xi^s) \;\geq\; (h(\xi^s) - T(\xi^s)x)^T \bar{\pi}^s \tag{3}$$

Summing over the scenarios, we obtain the valid subgradient inequality:

$$Q(x) \;\geq\; \sum_{s=1}^{S} p_s (h(\xi^s) - T(\xi^s)x)^T \bar{\pi}^s \;=\; Q(\bar{x}) \;-\; \sum_{s=1}^{S} p_s (T(\xi^s)^T \bar{\pi}^s)^T (x - \bar{x}) \tag{4}$$

2.2 The Master Problem

We replace $Q(x)$ by the scalar variable t in the original first stage problem, and then require t to be no smaller than the piecewise linear approximation to $Q(x)$. Thus, at iteration K, the Master Problem has the form

$$\begin{array}{llcccll}
\min_{x,t} & c^T x & + & t & & & \\
\text{subject to} & Ax & & & = & b & \\
& (\zeta^k)^T(x - x^k) & + & t & \geq & Q(x^k) & \text{for } k = 1, 2, \ldots, K-1 \\
& x & & & \geq & 0 &
\end{array} \tag{5}$$

where x^k is the solution to the kth Master Problem, and then ζ^k comes from the subgradient inequality (4). Thus,

$$\zeta^k = \sum_{s=1}^{S} p_s \, T(\xi^s)^T \pi_k^s$$

where π_k^s is the dual solution to the subproblem for scenario s when $x = x^k$. This is the **aggregated** form of the Master Problem. We can also develop a **disaggregated** form, with separate t variables for each scenario.

3 Extending the L-shaped method to integer second stage variables

If we can use polyhedral theory to represent the second stage problem as a linear program then we can use Benders decomposition. Thus, ideally, we want to find linear constraints

$$Gy \geq p(\xi) - M(\xi)x$$

so that

$$\begin{array}{l}
\{y \in \mathcal{Y} : Wy = h(\xi) - T(\xi)x\} \\
\quad = \{y \geq 0 : Wy = h(\xi) - T(\xi)x, \, Gy \geq p(\xi) - M(\xi)x\} \quad \text{for each scenario } \xi.
\end{array}$$

The polyhedral approximation is constructed iteratively. When the first stage variables are binary, Gomory cutting planes can be used. Let y_k be a basic variable that is non-integral in the optimal solution to a subproblem corresponding to one of the scenarios when $x = \bar{x}$, a binary vector. Let R denote the set of nonbasic variables y_j in the optimal solution. We can write the corresponding row of the simplex tableau for the second-stage problem as follows:

$$y_k + \sum_{j \in R} \gamma_j y_j + \sum_{i:\bar{x}_i = 0} \nu_i(\xi) x_i + \sum_{i:\bar{x}_i = 1} \nu_i(\xi)(1 - x_i) = \beta_k(\xi).$$

A Gomory cutting plane can be derived from this equality. This cut is valid for any x and for any y that is feasible in the second stage problem for this realization of ξ.

For the cut to be valid for any x, we need the x to be constrained to be binary. The construction doesn't work for more general integer variables. We need to exploit the binary property to argue either that x_i can only increase (if $\bar{x}_i = 0$), or that $1 - x_i$ can only increase (if $1 - \bar{x}_i = 0$).

What about the master problem?

As we add constraints to the second stage problem, we add variables to its dual. Our subgradient cuts in the master problem depend on dual feasibility, not necessarily optimality. If the new dual variables are given the value 0, all the old dual feasible solutions are still feasible. Hence, all the generated master problem cuts remain valid in the master problem.

Convergence

The arguments for finite convergence of Gomory's cutting plane algorithm for integer programming extend out to the SIP setting also.