Stochastic Programming Introduction

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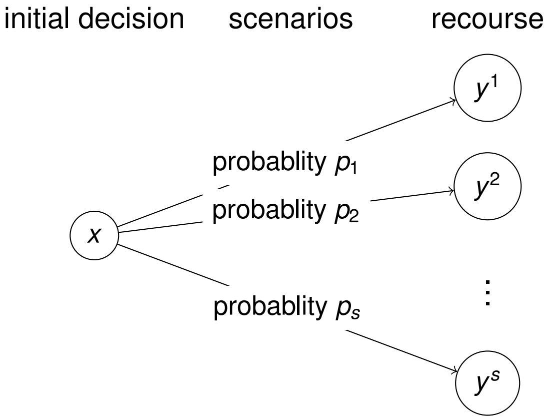
Outline

Introduction

In a two-stage stochastic program:

- we make an initial decision x, then
- a random scenario ξ occurs with probability p, and
- we make another (*recourse*) decision y.



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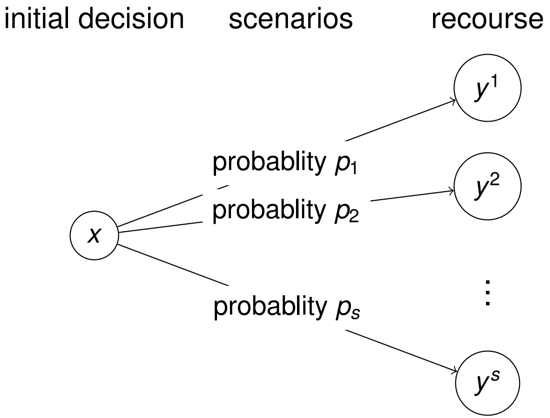

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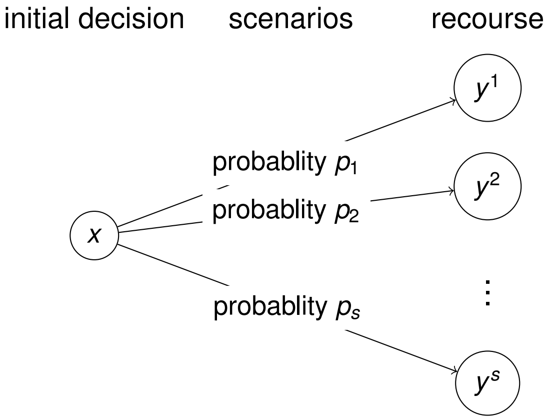


Objective function

The standard objective is to minimize the expected cost.

Other objective functions can be used.

For example, in robust optimization we minimize the worst scenario.

In a CVaR approach we minimize the average cost of the worst few scenarios.

The scenario that is "worst" depends on the first stage decision x.

For recent surveys see [1, 2].

Outline

Stochastic server location

We have n_1 possible server locations and m possible customers.

We pay a fixed cost c_i for choosing to open a server at location i.

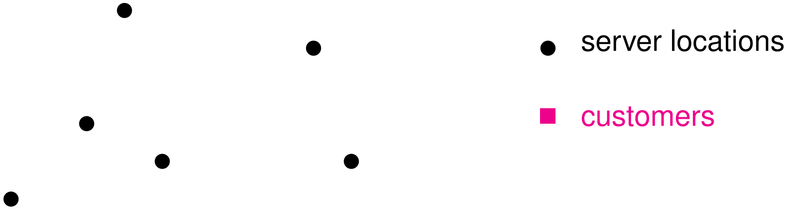
We must place at least one server, and no more than r servers.

We have to locate the servers before we know the integral demand $d_j(\xi)$ of the customers j.

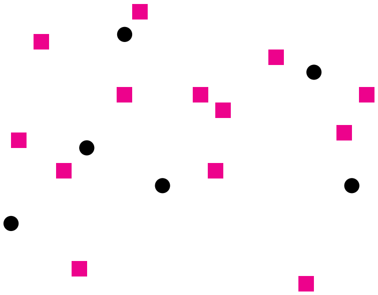
We assume any server can serve any customer, and the profit for each unit of demand of customer j met from server i is g_{ij}.

The servers have soft capacities w_i for each server i, in that we must pay a penalty g_{i0} per unit if the demand at server i is greater than its capacity w_i.

Server location

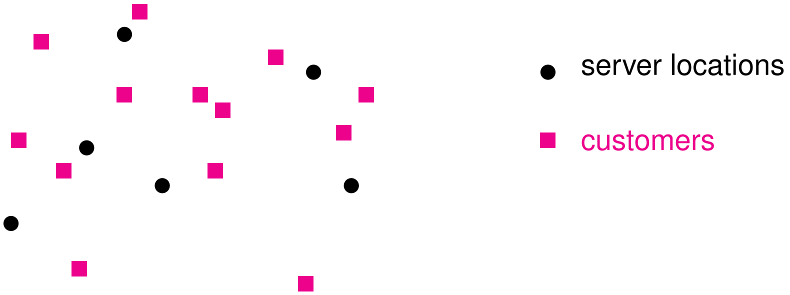


Server location



- server locations
- customers

Server location



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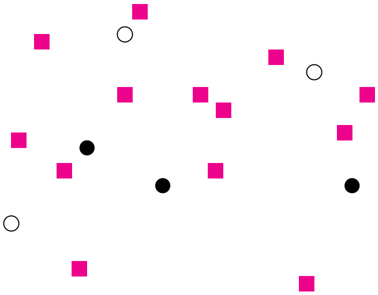


- server locations
- customers

First stage model

Let x_i denote the binary variable indicating whether or not we place a server at location i for each i.

We can model the first stage problem:

$$\begin{array}{lrcl} \min_x & c^T x + \mathbb{E}(x,\xi) & & \\ \text{subject to} & e^T x & \leq & r \\ & e^T x & \geq & 1 \\ & x & \in & \mathbb{B}^{n_1} \end{array}$$

$= \sum_s p_s \times$ (value of scenario)

p_s: prob of scenario

where e denotes the vector of ones.

Second stage model

For a given realization ξ, we introduce second stage variables y_{ij} to represent the amount of demand of customer j that is met by server i, and z_i to denote the shortfall at server i.

The second stage problem can be written

$$
\begin{array}{lrcrcl}
\min_{y,z} & \sum_i g_{i0} z_i & - & \sum_i \sum_j g_{ij} y_{ij} & & \\
\text{subject to} & -z_i & + & \sum_j y_{ij} & \leq & w_i x_i \ \text{ for each server } i \\
 & & & \sum_i y_{ij} & = & d_j \ \text{ for each customer } j \\
 & & & z, \ y & & \text{integer, nonnegative}
\end{array}
$$

$$z_i \leq D x_i$$

Outline

General formulation

mixed integer

We will focus on **stochastic two stage ~~linear~~ programs with recourse.** The general formulation can be written

$$\begin{array}{lrcl} \min_x & c^T x & + & \mathbb{E}_\xi\left[Q(x,\xi)\right] \\ \text{subject to} & Ax & = & b \\ & x & \in & \mathcal{X} \end{array}$$

where the first stage decisions are $x \in \mathcal{X} \subseteq \boldsymbol{R}^{n_1}$, the constraint matrix A is $m_1 \times n_1$, $b \in \boldsymbol{R}^{m_1}$, $c \in \boldsymbol{R}^{n_1}$.

Further, ξ is the uncertainty, and $Q(x,\xi)$ is the cost of the recourse decision when the first stage decision is x and the uncertainty is ξ.

Thus, $Q(x,\xi)$ is the second stage cost. We take the expectation of the second stage cost over all scenarios ξ.

Second stage cost

The second stage cost

$$
\begin{array}{rlrcl}
Q(x,\xi) = & \min_y & q^T y & & \\
& \text{subject to} & Wy & = & h(\xi) - T(\xi)x \\
& & y & \in & \mathcal{Y}
\end{array}
$$

where $y \in \mathcal{Y} \subseteq \boldsymbol{R}^{n_2}$, W is a fixed $m_2 \times n_2$ matrix, the right hand side $h(\xi) \in \boldsymbol{R}^{m_2}$ depends on the uncertainty ξ, and the $m_2 \times n_2$ technology matrix $T(\xi)$ also depends on ξ.

Note that the second stage optimization is over y, with x taken as a parameter.

Integrality

The sets $\mathcal{X}$ and $\mathcal{Y}$ impose nonnegativity, and discrete, binary, or continuous restrictions on the first and second-stage variables, respectively.

Explicit MIP formulation

Assume we have a finite number of scenarios $s = 1, \ldots, S$, each with probability p_s.

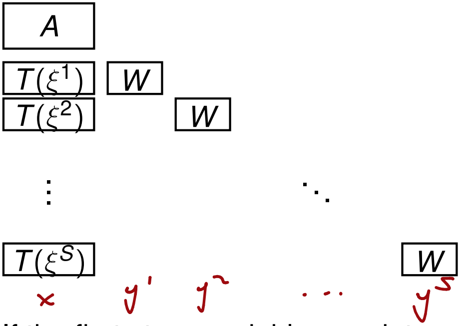
We introduce separate copies y^s of y for each scenario s.

The complete problem can then be written as an explicit ~~linear~~ mixed integer program:

$$
\begin{array}{llcrcll}
\min_{x,y} & c^T x & + & \sum_{s=1}^{S} p_s q^T y^s & & & \\
\text{subject to} & Ax & & & = & b & \\
& T(\xi^s)x & + & Wy^s & = & h(\xi^s) & s = 1, \ldots, S \\
& x & & & \in & \mathcal{X} & \\
& & & y^s & \in & \mathcal{Y}, & s = 1, \ldots, S
\end{array}
$$

Constraint matrix structure

The primal constraint matrix has the structure



If the first stage variables are integral and the second-stage variables are continuous, we can use Benders decomposition.

The second stage subproblems are separable, with a different subproblem for each scenario.

This is known as the *L-shaped* method in the stochastic programming literature.

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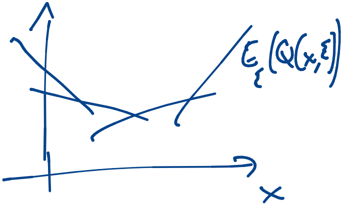
Theoretical considerations



When the second stage variables are all continuous, the expectation function $\mathbb{E}_\xi\left[Q(x,\xi)\right]$ is continuous and convex.

However, if some of the second stage variables are required to be integral, this function can be discontinuous.

Example

A simple example with just one scenario, with x and y being scalar variables:

$$\begin{array}{llll} \min_{x,y} & 3x & + & 4y \\ \text{subject to} & x & \leq & 6 \\ & x & \geq & 0, \ \text{integer} \end{array}$$

where y solves the subproblem

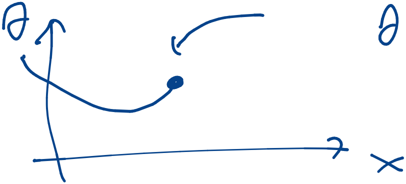
$$\begin{array}{llll} \min_{y} & y & & \\ \text{subject to} & 2y & = & 6 - x \\ & y & \geq & 0, \ \text{integer} \end{array}$$

The feasible solutions require x be even.
If x is odd then the subproblem is infeasible, so we can say that such a solution x has value $+\infty$.

Continuity

It is common to assume *complete recourse*:
the subproblem is feasible for any choice of first-stage variable that satisfies the first-stage constraints.

Under this assumption and some other assumptions, it can be shown that $\mathbb{E}_{\xi}\left[Q(x,\xi)\right]$ is well-defined, real valued, and lower semicontinuous, although it may still not be convex or even continuous.



$$Q = \begin{cases} 2+(x-1)^2 & x \le 2 \\ \ldots & x > 2 \end{cases}$$

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References

S. Ahmed.
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