Stochastic Programming Introduction

John E. Mitchell

Department of Mathematical Sciences
RPI, Troy, NY 12180 USA

April 2019
Outline

1. Introduction
2. A server location example
3. General formulation
4. Theoretical considerations
5. References
Introduction
In a two-stage stochastic program:
- we make an initial decision $x$, then
- a random scenario $\xi$ occurs with probability $p$, and
- we make another (recourse) decision $y$.

Initial decision scenarios recourse

- Probability $p_1$
- Probability $p_2$
- Probability $p_s$

$y^1$  $y^2$  $\vdots$  $y^s$
In a two-stage stochastic program:
- we make an initial decision $x$, then
- a random scenario $\xi$ occurs with probability $p$, and
- we make another (recourse) decision $y$.

initial decision     scenarios     recourse

- $x$ leads to scenarios with probabilities $p_1, p_2, \ldots, p_s$
  - $y_1$ with probability $p_1$
  - $y_2$ with probability $p_2$
  - $y_s$ with probability $p_s$
In a two-stage stochastic program:
- we make an initial decision $x$, then
- a random scenario $\xi$ occurs with probability $p$, and
- we make another (recourse) decision $y$.

initial decision  scenarios  recourse

$\begin{align*}
x & \rightarrow y^1 \\
 & \quad \text{probability } p_1 \\
 & \rightarrow y^2 \\
 & \quad \text{probability } p_2 \\
 & \rightarrow \cdots \\
 & \rightarrow y^s \\
 & \quad \text{probability } p_s
\end{align*}$
Objective function

The standard objective is to minimize the expected cost.

Other objective functions can be used.

For example, in robust optimization we minimize the worst scenario. In a CVaR approach we minimize the average cost of the worst few scenarios.

The scenario that is “worst” depends on the first stage decision $x$.

For recent surveys see [1, 2].
Outline

1. Introduction
2. A server location example
3. General formulation
4. Theoretical considerations
5. References
Stochastic server location

We have $n_1$ possible server locations and $m$ possible customers.
We pay a fixed cost $c_i$ for choosing to open a server at location $i$.
We must place at least one server, and no more than $r$ servers.
We have to locate the servers before we know the integral demand $d_j(\xi)$ of the customers $j$.

We assume any server can serve any customer, and the profit for each unit of demand of customer $j$ met from server $i$ is $g_{ij}$.

The servers have soft capacities $w_i$ for each server $i$, in that we must pay a penalty $g_{i0}$ per unit if the demand at server $i$ is greater than its capacity $w_i$. 
Server location

- server locations
- customers
Server location

- server locations
- customers
Server location

- Server locations
- Customers
Server location

*server locations*

*customers*
First stage model

Let $x_i$ denote the binary variable indicating whether or not we place a server at location $i$ for each $i$.

We can model the first stage problem:

\[
\min_x \quad c^T x + \mathbb{E}(x, \xi) \\
\text{subject to} \quad e^T x \leq r \\
\quad e^T x \geq 1 \\
\quad x \in \mathbb{B}^{n_1}
\]

where $e$ denotes the vector of ones.
Second stage model

For a given realization $\xi$, we introduce second stage variables $y_{ij}$ to represent the amount of demand of customer $j$ that is met by server $i$, and $z_i$ to denote the shortfall at server $i$.

The second stage problem can be written

$$\min_{y,z} \quad \sum_i g_i z_i - \sum_i \sum_j g_{ij} y_{ij}$$

subject to

$$-z_i + \sum_j y_{ij} \leq w_i x_i \text{ for each server } i$$

$$\sum_i y_{ij} = d_j \text{ for each customer } j$$

$z, y$ integer, nonnegative
Outline

1 Introduction
2 A server location example
3 General formulation
4 Theoretical considerations
5 References
General formulation

We will focus on stochastic two stage mixed integer programs with recourse. The general formulation can be written

$$\min_x \ c^T x + \mathbb{E}_\xi [Q(x, \xi)]$$
subject to $\ A x = b$
$x \in \mathcal{X}$

where the first stage decisions are $x \in \mathcal{X} \subseteq \mathbb{R}^{n_1}$, the constraint matrix $A$ is $m_1 \times n_1$, $b \in \mathbb{R}^{m_1}$, $c \in \mathbb{R}^{n_1}$.

Further, $\xi$ is the uncertainty, and $Q(x, \xi)$ is the cost of the recourse decision when the first stage decision is $x$ and the uncertainty is $\xi$.

Thus, $Q(x, \xi)$ is the second stage cost. We take the expectation of the second stage cost over all scenarios $\xi$. 
Second stage cost

The second stage cost

\[ Q(x, \xi) = \min_y \quad q^T y \quad \text{subject to} \quad W y = h(\xi) - T(\xi) x \]

where \( y \in \mathcal{Y} \subseteq \mathbb{R}^{n_2} \), \( W \) is a fixed \( m_2 \times n_2 \) matrix, the right hand side \( h(\xi) \in \mathbb{R}^{m_2} \) depends on the uncertainty \( \xi \), and the \( m_2 \times n_2 \) technology matrix \( T(\xi) \) also depends on \( \xi \).

Note that the second stage optimization is over \( y \), with \( x \) taken as a parameter.
Integrality

The sets $\mathcal{X}$ and $\mathcal{Y}$ impose nonnegativity, and discrete, binary, or continuous restrictions on the first and second-stage variables, respectively.
Explicit MIP formulation

Assume we have a finite number of scenarios \( s = 1, \ldots, S \), each with probability \( p_s \).

We introduce separate copies \( y^s \) of \( y \) for each scenario \( s \).

The complete problem can then be written as an explicit mixed integer program:

\[
\begin{align*}
\min_{x,y} & \quad c^T x + \sum_{s=1}^{S} p_s q^T y^s \\
\text{subject to} & \quad Ax + T(\xi^s)x + Wy^s &= b \\
& \quad W y^s = h(\xi^s) \quad s = 1, \ldots, S \\
& \quad x \in X \\
& \quad y^s \in Y, \quad s = 1, \ldots, S
\end{align*}
\]
Constraint matrix structure

The primal constraint matrix has the structure

\[
\begin{bmatrix}
A \\
T(\xi^1) & W \\
T(\xi^2) & W \\
\vdots & \vdots \\
T(\xi^S) & W
\end{bmatrix}
\]

If the first stage variables are integral and the second-stage variables are continuous, we can use Benders decomposition.

The second stage subproblems are separable, with a different subproblem for each scenario.

This is known as the \textit{L-shaped} method in the stochastic programming literature.
Outline

1. Introduction
2. A server location example
3. General formulation
4. Theoretical considerations
5. References
Theoretical considerations

When the second stage variables are all continuous, the expectation function $\mathbb{E}_\xi [Q(x, \xi)]$ is continuous and convex.

However, if some of the second stage variables are required to be integral, this function can be discontinuous.
Example

A simple example with just one scenario, with $x$ and $y$ being scalar variables:

$$\min_{x,y} \quad 3x + 4y$$
subject to
$$x \leq 6$$
$$x \geq 0, \text{ integer}$$

where $y$ solves the subproblem

$$\min_y \quad y$$
subject to
$$2y = 6 - x$$
$$y \geq 0, \text{ integer}$$

The feasible solutions require $x$ be even. If $x$ is odd then the subproblem is infeasible, so we can say that such a solution $x$ has value $+\infty$.
It is common to assume *complete recourse*: the subproblem is feasible for any choice of first-stage variable that satisfies the first-stage constraints.

Under this assumption and some other assumptions, it can be shown that $\mathbb{E}_\xi [Q(x, \xi)]$ is well-defined, real valued, and lower semicontinuous, although it may still not be convex or even continuous.
Outline

1. Introduction
2. A server location example
3. General formulation
4. Theoretical considerations
5. References
References

S. Ahmed.
Two stage stochastic integer programming.

S. Küçükyavuz and S. Sen.
An introduction to two-stage stochastic mixed-integer programming.
S. Ahmed.
Two stage stochastic integer programming.

S. Küçükyavuz and S. Sen.
An introduction to two-stage stochastic mixed-integer programming.