Stochastic Programming Introduction

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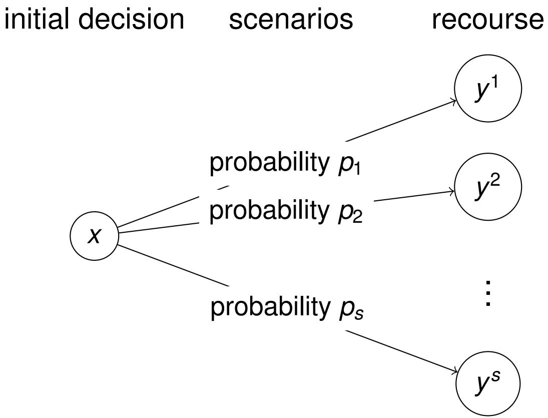
Outline

Introduction

In a two-stage stochastic program:

- we make an initial decision x, then
- a random scenario ξ occurs with probability p, and
- we make another (*recourse*) decision y.



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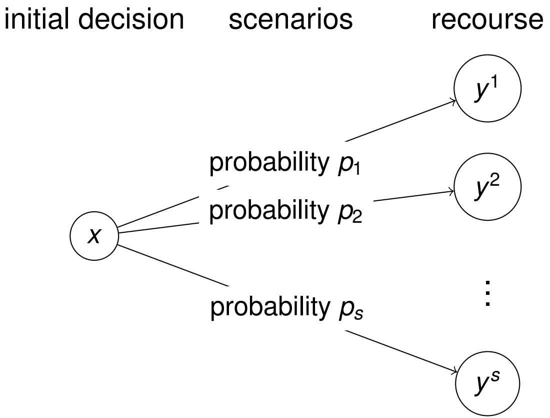

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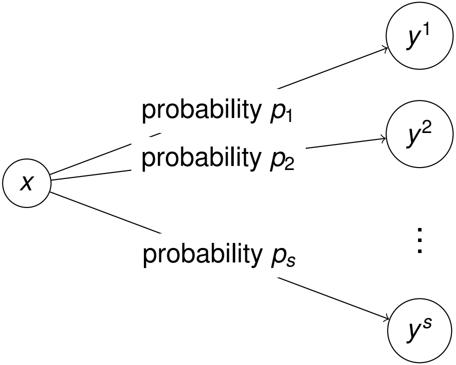
initial decision scenarios recourse



Objective function

The standard objective is to minimize the expected cost.

Other objective functions can be used.

For example, in robust optimization we minimize the worst scenario.

In a CVaR approach we minimize the average cost of the worst few scenarios.

The scenario that is “worst” depends on the first stage decision x.

For recent surveys see [1, 2].

Outline

Stochastic server location

We have n_1 possible server locations and m possible customers.

We pay a fixed cost c_i for choosing to open a server at location i.

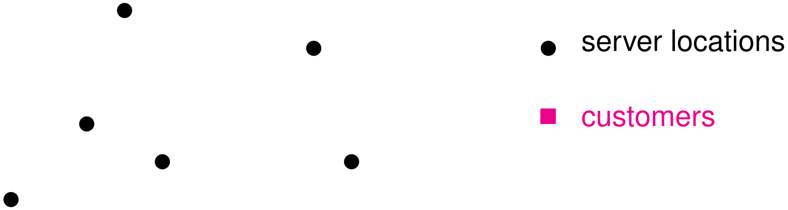
We must place at least one server, and no more than r servers.

We have to locate the servers before we know the integral demand $d_j(\xi)$ of the customers j.

We assume any server can serve any customer, and the profit for each unit of demand of customer j met from server i is g_{ij}.

The servers have soft capacities w_i for each server i, in that we must pay a penalty g_{i0} per unit if the demand at server i is greater than its capacity w_i.

Server location



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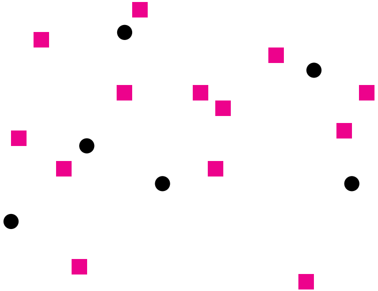
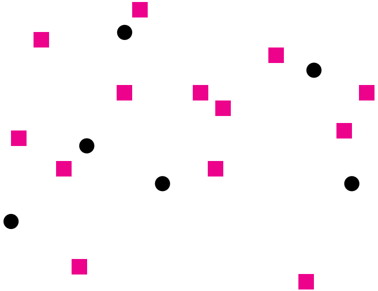


● server locations

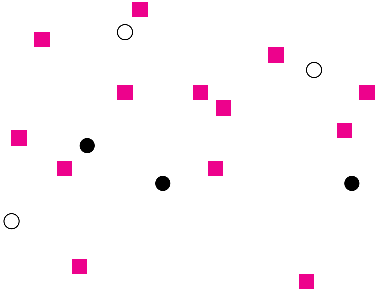
■ customers

Server location



- server locations
- customers

Server location



- server locations
- customers

First stage model

Let x_i denote the binary variable indicating whether or not we place a server at location i for each i.

We can model the first stage problem:

$$\begin{array}{lrcl} \min_x & c^T x + \mathbb{E}(x,\xi) & & \\ \text{subject to} & e^T x & \leq & r \\ & e^T x & \geq & 1 \\ & x & \in & \mathbb{B}^{n_1} \end{array}$$

where e denotes the vector of ones.

Second stage model

For a given realization ξ, we introduce second stage variables y_{ij} to represent the amount of demand of customer j that is met by server i, and z_i to denote the shortfall at server i.

The second stage problem can be written

$$
\begin{array}{llcrcl}
\min_{y,z} & \sum_i g_{i0} z_i & - & \sum_i \sum_j g_{ij} y_{ij} & & \\
\text{subject to} & -z_i & + & \sum_j y_{ij} & \leq & w_i x_i \text{ for each server } i \\
 & & & \sum_i y_{ij} & = & d_j \text{ for each customer } j \\
 & & & z,\ y & & \text{integer, nonnegative}
\end{array}
$$

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General formulation

We will focus on **stochastic two stage mixed integer programs with recourse.** The general formulation can be written

$$\begin{array}{llll} \min_x & c^T x & + & \mathbb{E}_\xi \left[Q(x,\xi) \right] \\ \text{subject to} & Ax & = & b \\ & x & \in & \mathcal{X} \end{array}$$

where the first stage decisions are $x \in \mathcal{X} \subseteq \boldsymbol{R}^{n_1}$, the constraint matrix A is $m_1 \times n_1$, $b \in \boldsymbol{R}^{m_1}$, $c \in \boldsymbol{R}^{n_1}$.

Further, ξ is the uncertainty, and $Q(x,\xi)$ is the cost of the recourse decision when the first stage decision is x and the uncertainty is ξ.

Thus, $Q(x,\xi)$ is the second stage cost. We take the expectation of the second stage cost over all scenarios ξ.

Second stage cost

The second stage cost

$$
\begin{array}{rlrcl}
Q(x,\xi) = & \min_y & q^T y & & \\
& \text{subject to} & Wy & = & h(\xi) - T(\xi)x \\
& & y & \in & \mathcal{Y}
\end{array}
$$

where $y \in \mathcal{Y} \subseteq \boldsymbol{R}^{n_2}$, W is a fixed $m_2 \times n_2$ matrix, the right hand side $h(\xi) \in \boldsymbol{R}^{m_2}$ depends on the uncertainty ξ, and the $m_2 \times n_2$ technology matrix $T(\xi)$ also depends on ξ.

Note that the second stage optimization is over y, with x taken as a parameter.

Integrality

The sets $\mathcal{X}$ and $\mathcal{Y}$ impose nonnegativity, and discrete, binary, or continuous restrictions on the first and second-stage variables, respectively.

Explicit MIP formulation

Assume we have a finite number of scenarios $s = 1, \ldots, S$, each with probability p_s.

We introduce separate copies y^s of y for each scenario s.

The complete problem can then be written as an explicit mixed integer program:

$$
\begin{array}{llcrcll}
\min_{x,y} & c^T x & + & \sum_{s=1}^{S} p_s q^T y^s & & & \\
\text{subject to} & Ax & & & = & b & \\
 & T(\xi^s)x & + & Wy^s & = & h(\xi^s) & s = 1, \ldots, S \\
 & x & & & \in & \mathcal{X} & \\
 & & & y^s & \in & \mathcal{Y}, & s = 1, \ldots, S
\end{array}
$$

Constraint matrix structure

The primal constraint matrix has the structure

$$
\begin{array}{lllll}
\boxed{A} & & & & \\
\boxed{T(\xi^1)} & \boxed{W} & & & \\
\boxed{T(\xi^2)} & & \boxed{W} & & \\
\vdots & & & \ddots & \\
\boxed{T(\xi^S)} & & & & \boxed{W}
\end{array}
$$

If the first stage variables are integral and the second-stage variables are continuous, we can use Benders decomposition.

The second stage subproblems are separable, with a different subproblem for each scenario.

This is known as the *L*-shaped method in the stochastic programming literature.

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Theoretical considerations

When the second stage variables are all continuous, the expectation function $\mathbb{E}_\xi\left[Q(x,\xi)\right]$ is continuous and convex.

However, if some of the second stage variables are required to be integral, this function can be discontinuous.

Example

A simple example with just one scenario, with x and y being scalar variables:

$$\begin{array}{lrcl} \min_{x,y} & 3x & + & 4y \\ \text{subject to} & x & \leq & 6 \\ & x & \geq & 0, \ \text{integer} \end{array}$$

where y solves the subproblem

$$\begin{array}{lrcl} \min_{y} & y & & \\ \text{subject to} & 2y & = & 6 - x \\ & y & \geq & 0, \ \text{integer} \end{array}$$

The feasible solutions require x be even.
If x is odd then the subproblem is infeasible, so we can say that such a solution x has value $+\infty$.

Continuity

It is common to assume *complete recourse*:
the subproblem is feasible for any choice of first-stage variable that satisfies the first-stage constraints.

Under this assumption and some other assumptions, it can be shown that $\mathbb{E}_\xi\left[Q(x,\xi)\right]$ is well-defined, real valued, and lower semicontinuous, although it may still not be convex or even continuous.

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References

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Two stage stochastic integer programming.
In T. Choran, editor, *Encyclopedia of Optimization*. Wiley, 2011.

S. Küçükyavuz and S. Sen.
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