MATP6620/ISYE6760 Integer and Combinatorial Optimization

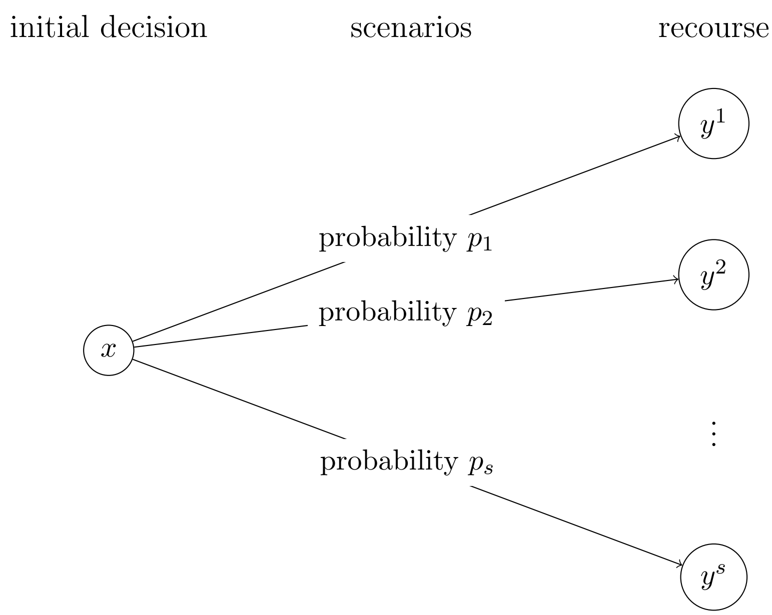
Stochastic Programming Introduction

John E. Mitchell

1 Introduction

In a two-stage stochastic program:

- we make an initial decision x, then
- a random scenario ξ occurs with probability p, and
- we make another (*recourse*) decision y.



The standard objective is to minimize the expected cost. Other objective functions can be used. For example, in robust optimization we minimize the worst scenario and in a CVaR approach we minimize the average cost of the worst few scenarios. The scenario that is "worst" depends on the first stage decision x. For recent surveys see [1, 2].

We will focus on **stochastic two stage integer linear programs with recourse**. The general formulation can be written

$$\begin{array}{lrcl} \min_x & c^T x & + & \mathbb{E}_\xi \left[Q(x,\xi) \right] \\ \text{subject to} & Ax & = & b \\ & x & \in & \mathcal{X} \end{array}$$

where the first stage decisions are $x \in \mathcal{X} \subseteq \mathbb{R}^{n_1}$, the constraint matrix A is $m_1 \times n_1$, $b \in \mathbb{R}^{m_1}$, $c \in \mathbb{R}^{n_1}$, ξ is the uncertainty, and $Q(x, \xi)$ is the cost of the recourse decision when the first stage decision is x and the uncertainty is ξ. Thus, $Q(x, \xi)$ is the second stage cost. We take the expectation of the second stage cost over all scenarios ξ.

The second stage cost

$$\begin{array}{rlrcl} Q(x,\xi) = & \min_y & q^T y & & \\ & \text{subject to} & Wy & = & h(\xi) - T(\xi)x \\ & & y & \in & \mathcal{Y} \end{array}$$

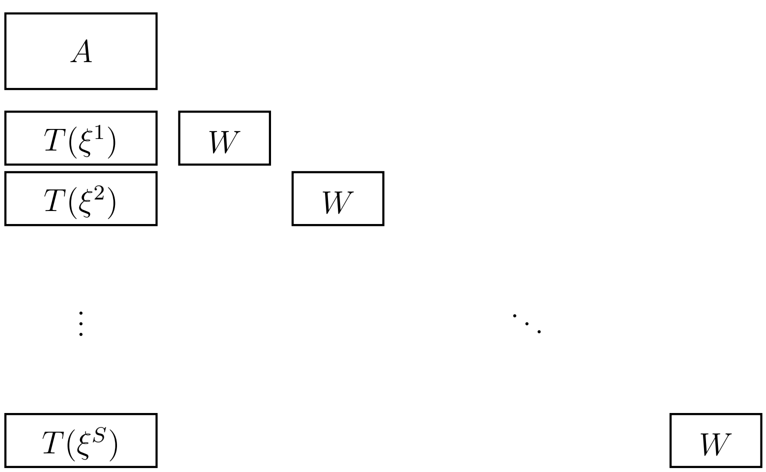
where $y \in \mathcal{Y} \subseteq \mathbb{R}^{n_2}$, W is a fixed $m_2 \times n_2$ matrix, the right hand side $h(\xi) \in \mathbb{R}^{m_2}$ depends on the uncertainty ξ, and the $m_2 \times n_2$ technology matrix $T(\xi)$ also depends on ξ. Note that the second stage optimization is over y, with x taken as a parameter.

The sets $\mathcal{X}$ and $\mathcal{Y}$ impose nonnegativity, and discrete, binary, or continuous restrictions on the first and second-stage variables, respectively.

Assume we have a finite number of scenarios $s = 1, \ldots, S$, each with probability p_s. We introduce separate copies y^s of y for each scenario s. The complete problem can then be written as an explicit mixed integer program:

$$\begin{array}{llclcll} \min_{x,y} & c^T x & + & \sum_{s=1}^S p_s q^T y^s & & & \\ \text{subject to} & Ax & & & = & b & \\ & T(\xi^s)x & + & Wy^s & = & h(\xi^s) & s = 1, \ldots, S \\ & x & & & \in & \mathcal{X} & \\ & & & y^s & \geq & \mathcal{Y}, & s = 1, \ldots, S \end{array}$$

The primal constraint matrix has the structure



This structure is amenable to decomposition approaches. For example, if the first stage variables are integral and the second-stage variables are continuous, we can use Benders decomposition. The second stage subproblems are separable, with a different subproblem for each scenario. This is known as the L-shaped method in the stochastic programming literature.

2 Theoretical considerations

When the second stage variables are all continuous, the expectation function $\mathbb{E}_\xi[Q(x,\xi)]$ is continuous and convex. However, if some of the second stage variables are required to be integral, this function can be discontinuous.

Example 1. *A simple example with just one scenario, with x and y being scalar variables:*

$$\begin{array}{lrcl} \min_{x,y} & 3x & + & 4y \\ \textit{subject to} & x & \leq & 6 \\ & x & \geq & 0, \ \textit{integer} \end{array}$$

where y solves the subproblem

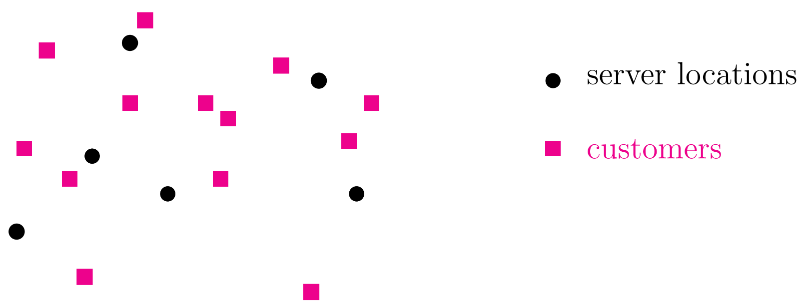
$$\begin{array}{lrcl} \min_{y} & y & & \\ \textit{subject to} & 2y & = & 6 - x \\ & y & \geq & 0, \ \textit{integer} \end{array}$$

The feasible solutions require x be even. If x is odd then the subproblem is infeasible, so we can say that such a solution x has value $+\infty$.

It is common to assume *complete recourse*; that is, the subproblem is feasible for any choice of first-stage variable that satisfies the first-stage constraints. Under this assumption and some other assumptions, it can be shown that $\mathbb{E}_\xi[Q(x,\xi)]$ is well-defined, real valued, and lower semicontinuous, although it may still not be convex or even continuous.

3 A server location example

We have n_1 possible server locations and m possible customers. We pay a fixed cost c_i for choosing to open a server at location i. We must place at least one server, and no more than r servers. We have to locate the servers before we know the integral demand $d_j(\xi)$ of the customers j. We assume any server can serve any customer, and the profit for each unit of demand of customer j met from server i is g_{ij}. The servers have soft capacities w_i for each server i, in that we must pay a penalty g_{i0} per unit if the demand at server i is greater than its capacity w_i.



Let x_i denote the binary variable indicating whether or not we place a server at location i for each i. We can model the first stage problem:

$$\begin{array}{lrcl} \min_x & c^T x & + & \mathbb{E}(x,\xi) \\ \text{subject to} & e^T x & \leq & r \\ & e^T x & \geq & 1 \\ & x & \in & \mathbb{B}^{n_1} \end{array}$$

where e denotes the vector of ones. For a given realization ξ, we introduce second stage variables y_{ij} to represent the amount of demand of customer j that is met by server i, and z_i to denote the shortfall at server i. The second stage problem can be written

$$\begin{array}{lrcrcll} \min_{y,z} & \sum_i g_{i0} z_i & - & \sum_i \sum_j g_{ij} y_{ij} & & & \\ \text{subject to} & -z_i & + & \sum_j y_{ij} & \leq & w_i x_i & \text{for each server } i \\ & & & \sum_i y_{ij} & = & d_j & \text{for each customer } j \\ & & & z,\, y & & \text{integer, nonnegative} & \end{array}$$

References

[1] S. Ahmed. Two stage stochastic integer programming. In T. Choran, editor, *Encyclopedia of Optimization*. Wiley, 2011.

[2] S. Küçükyavuz and S. Sen. An introduciton to two-stage stochastic mixed-integer programming. In R. Batta and J. Peng, editors, *TutORials in Operations Research: Leading Developments from INFORMS Communities*, pages 1–27. INFORMS, Catonsville, MD, 2017.