1 Introduction

In a two-stage stochastic program:

- we make an initial decision $x$, then
- a random scenario $\xi$ occurs with probability $p$, and
- we make another (recourse) decision $y$.

The standard objective is to minimize the expected cost. Other objective functions can be used. For example, in robust optimization we minimize the worst scenario and in a CVaR approach we minimize the average cost of the worst few scenarios. The scenario that is “worst” depends on the first stage decision $x$. For recent surveys see [1], [2].

We will focus on stochastic two stage integer linear programs with recourse. The general formulation can be written

$$\min_x \ c^T x + \mathbb{E}_\xi [Q(x, \xi)]$$
subject to

$$Ax = b$$
$$x \in \mathcal{X}$$
where the first stage decisions are $x \in \mathcal{X} \subseteq \mathbb{R}^{n_1}$, the constraint matrix $A$ is $m_1 \times n_1$, $b \in \mathbb{R}^{m_1}$, $c \in \mathbb{R}^{n_1}$, $\xi$ is the uncertainty, and $Q(x, \xi)$ is the cost of the recourse decision when the first stage decision is $x$ and the uncertainty is $\xi$. Thus, $Q(x, \xi)$ is the second stage cost. We take the expectation of the second stage cost over all scenarios $\xi$.

The second stage cost

$$Q(x, \xi) = \min_y q^T y$$

subject to

$$Wy = h(\xi) - T(\xi)x$$

where $y \in \mathcal{Y} \subseteq \mathbb{R}^{n_2}$, $W$ is a fixed $m_2 \times n_2$ matrix, the right hand side $h(\xi) \in \mathbb{R}^{m_2}$ depends on the uncertainty $\xi$, and the $m_2 \times n_2$ technology matrix $T(\xi)$ also depends on $\xi$. Note that the second stage optimization is over $y$, with $x$ taken as a parameter.

The sets $\mathcal{X}$ and $\mathcal{Y}$ impose nonnegativity, and discrete, binary, or continuous restrictions on the first and second-stage variables, respectively.

Assume we have a finite number of scenarios $s = 1, \ldots, S$, each with probability $p_s$. We introduce separate copies $y^s$ of $y$ for each scenario $s$. The complete problem can then be written as an explicit mixed integer program:

$$\min_{x, y^1, \ldots, y^S} c^T x + \sum_{s=1}^S p_s q^T y^s$$

subject to

$$Ax + T(\xi s)x = b$$

$$Wy^s = h(\xi s) \quad s = 1, \ldots, S$$

$$y^s \geq \mathcal{Y}, \quad s = 1, \ldots, S$$

The primal constraint matrix has the structure

\[
\begin{array}{c}
A \\
T(\xi^1) & W \\
T(\xi^2) & W \\
\vdots & \ddots \\
T(\xi^S) & W
\end{array}
\]

This structure is amenable to decomposition approaches. For example, if the first stage variables are integral and the second-stage variables are continuous, we can use Benders decomposition. The second stage subproblems are separable, with a different subproblem for each scenario. This is known as the $L$-shaped method in the stochastic programming literature.
2 Theoretical considerations

When the second stage variables are all continuous, the expectation function $\mathbb{E}_\xi [Q(x, \xi)]$ is continuous and convex. However, if some of the second stage variables are required to be integral, this function can be discontinuous.

Example 1. A simple example with just one scenario, with $x$ and $y$ being scalar variables:

$$\begin{align*}
\min_{x,y} & \quad 3x + 4y \\
\text{subject to} & \quad x \leq 6 \\
& \quad x \geq 0, \text{ integer}
\end{align*}$$

where $y$ solves the subproblem

$$\begin{align*}
\min_y & \quad y \\
\text{subject to} & \quad 2y = 6 - x \\
& \quad y \geq 0, \text{ integer}
\end{align*}$$

The feasible solutions require $x$ be even. If $x$ is odd then the subproblem is infeasible, so we can say that such a solution $x$ has value $+\infty$.

It is common to assume complete recourse; that is, the subproblem is feasible for any choice of first-stage variable that satisfies the first-stage constraints. Under this assumption and some other assumptions, it can be shown that $\mathbb{E}_\xi [Q(x, \xi)]$ is well-defined, real valued, and lower semicontinuous, although it may still not be convex or even continuous.
3 A server location example

We have $n_1$ possible server locations and $m$ possible customers. We pay a fixed cost $c_i$ for choosing to open a server at location $i$. We must place at least one server, and no more than $r$ servers. We have to locate the servers before we know the integral demand $d_j(\xi)$ of the customers $j$. We assume any server can serve any customer, and the profit for each unit of demand of customer $j$ met from server $i$ is $g_{ij}$. The servers have soft capacities $w_i$ for each server $i$, in that we must pay a penalty $g_{i0}$ per unit if the demand at server $i$ is greater than its capacity $w_i$.

Let $x_i$ denote the binary variable indicating whether or not we place a server at location $i$ for each $i$. We can model the first stage problem:

$$\min_x \quad c^T x + \mathbb{E}(x, \xi)$$

subject to

$$e^T x \leq r$$

$$e^T x \geq 1$$

$$x \in \mathbb{B}^{n_1}$$

where $e$ denotes the vector of ones. For a given realization $\xi$, we introduce second stage variables $y_{ij}$ to represent the amount of demand of customer $j$ that is met by server $i$, and $z_i$ to denote the shortfall at server $i$. The second stage problem can be written

$$\min_{y, z} \quad \sum_i g_{i0} z_i - \sum_i \sum_j g_{ij} y_{ij}$$

subject to

$$-z_i + \sum_j y_{ij} \leq w_i x_i \quad \text{for each server } i$$

$$\sum_j y_{ij} = d_j \quad \text{for each customer } j$$

$$z, y \quad \text{integer, nonnegative}$$

References
